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Noha EL-Hossiny Abu EL-Saud Abd Al-Rahman

Department of Statistics, Faculty of Commerce (Girls' Branch), Al-Azhar University, Tafahna Al-Ashraf, Egypt.

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*Corresponding author: : *noha@azhar.edu.eg*

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Department of Statistics, Faculty of Commerce (Girls' Branch), Al-Azhar University, Tafahna Al-Ashraf, Egypt.

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Abstract:

Many natural events with an S-shaped sigmoidal curve can be well described using sigmoid growth models. Sigmoid growth models are considered one of the most important and most widely used non-linear models in describing the most natural phenomena that have a sigmoidal growth curve in several disciplines such as the physical, chemical, biological, and social sciences. The purpose of the article is to suggested two new sigmoid growth models using two different techniques based on the Gompertz Exponential (GoE) distribution and comparison between them. The parameters of the suggested models are estimated using the maximum likelihood estimation technique. Through a Monte Carlo simulation and application utilizing Egypt's external debt data, the effectiveness of the newly presented models is examined and contrasted with some existing sigmoid growth such as Gompertz, and exponential models to explain the growth. Results showed that the recently suggested Transmuted Gompertz Exponential sigmoid growth model, and Gompertz Exponential sigmoid growth model are better than to the other models.

Keywords: External Debt; Sigmoid growth model; non-linear regression model; Gompertz model, Exponential model; Gompertz Exponential distribution; Maximum likelihood.

المستخلص:

يمكن وصف العديد من الأحداث الطبيعية ذات المنحنى السيني على شكل حرف S بشكل جيد باستخدام نماذج النمو السيني. تعتبر نماذج النمو السيني واحدة من أهم النماذج غير الخطية وأكثرها استخدامًا في وصف العديد من الظواهر الطبيعية التي لها منحنى نمو سيني في العديد من التخصصات مثل العلوم الفيزيائية والكيميائية والبيولوجية والاجتماعية. الغرض من هذه المقالة هو إقتراح نموذجين جديدين للنمو السيني بإستخدام طريقتين مختلفتين تعتمدان على توزيع غومبيرتز الأسي (GoE) والمقارنه بينهم. يتم تقدير معلمات النماذج المقترحة بإستخدام طريقة تقدير الإمكان الأعظم. من خلال محاكاة مونت كارلو والتطبيق على بيانات الدين الخارجى لمصر ، يتم فحص فعالية النماذج المقدمة حديثًا ومقارنتها ببعض نماذج النمو السيني الكلاسيكية مثل غومبيرتز، الأسي وذلك لتفسير النمو. أظهرت النتائج أن النماذج المقترحه، نموذج النمو السينى غومبيرتز الأسي المحول ، نموذج النمو السينى غومبيرتز الأسي أفضل من النماذج الأخرى.

الكلمات المفتاحية: الدين الخارجي، نموذج النمو السيني؛ نموج الانحدار الغير خطى ؛ توزيع غومبيرتز ؛ توزيع الأسي؛ توزيع غومبيرتز الأسي؛ الإمكان الأعظم .

1 Introduction

Non-linear regression models are frequently used to examine the influence of explanatory variables on a response variable across various fields, including physical, chemical, biological, and social sciences. The main reason for selecting a non-linear model often stems from existing knowledge that a specific process follows a known non-linear pattern. In such instances, the parameters can have significant and relevant connotations. Additionally, a secondary reason for opting for a non-linear model is the quest for simplicity; a well-chosen non-linear model can need much less parameters than multinomial model when analyzing the same dataset. There are several types of non-linear models, including the exponential decay model, segmented polynomial models, a two-term exponential model, inverse polynomial model, toxicity studies model, probit and logit models, as well as growth models.

One of the most important non-linear models is growth model; it has evolved over an extended period of time. As time passed, the primary emphasis switched from developing families of curves to depicting trends or typical behaviors. Sigmoid growth models are frequently used to represent plant weight, height, and index of leaf area, time dependent seed germination. More specifically, sigmoid functions are particularly interesting in abstract fields, the sigmoid function is relevant to the differential equations, formulas based on cumulative distribution, transmuted function, family of single sigmoid functions, and Hausdorff approximations. Sigmoid functions have a broad range of applications in physics, engineering, life and social sciences, including population dynamics, finance, signal and image processing, artificial neural networks, antenna feeding techniques, and insurance. Sigmoid growth models are non-linear regression models that have been used in a variety of domains with numerous notations and parameterizations. Some researches introduced and analysis non-linear regression models such as: [Ratkowsky (1983), Fekedulegn *et al*. (1999), Ritz and Streibig (2008), Archontoulis and Miguez (2015)].

Many researches have used, suggested growth models and analyzed various growth phenomena. For example, Carrillo and González (2002) introduced a new approach to modelling sigmoidal curves. Müller *et al.* (2006) used of a new sigmoid growth equation in growth models. Szabelska *et al.* (2010) presented five growth models: Exponential, Weibull , Logistic, Log-logistic and Gompertz. Goshu and Koya (2013) introduced logistic, generalized logistic, Richards, Von Bertalanffy, Brody, Gompertz, Weibull, generalized Weibull, and Monomolecular models. Mahanta and Borah (2014) discussed a few specific characteristics of three Weibull sigmoid growth models from a forestry perspective, while Panik (2014) introduced some of the more common parametric growth models such as Logistic, Gompertz, Weibull, Negative Exponential, and Von Bertalanffy. Sedmak and Scheer (2015) presented the features and short term forecast accuracy of the mathematical model of sigmoid time-determinate growth. Kim *et al.* (2017) suggested a growth model that includes the impact of water temperature on the development in the von Bertalanffy growth model. Fernandes *et al*. (2017) used the Logistic and Gompertz growth models for analyzing the growth pattern of coffee berries, while Souza *et al.* (2017) studied the

growth models: Gompertz, Logistic, Brody, and von Bertalanffy models in analyzing the cross section data for the weight of living things
of Mangalarga Marchador horses, also, Tjørve and Tjørve (2017) used of Gompertz models in growth analyses, introduced new Gompertz model approach. Amarti (2018) introduced the logistic growth model with the Allee effect for describing the rise of population numbers, Ghaderi-Zefrehel *et al*. (2018) studied some general non-linear growth models such as Gompertz, von Bertalanffy, Logistic, and Brody coupled with multilevel modelling to explore the typical development of Iranian Lori Bakhtiari sheep, while Ribeiro *et al*. (2018) explained the Gompertz and Logistic models were used to measure the develop and growth of Asian pear fruit over time based on diameter, length, and fresh weight. Cao *et al*. (2019) presented a new sigmoid growth model to describe the development of plants and animals when the growth rate curve is asymmetric, Ukalska and Jastrzebowski (2019) studied the dynamics of the epicotyls emergence of oak using the Logistic, Gompertz, and Richards models, also, Zardin *et al.* (2019) studied the growth curves by Gompertz non-linear regression model. Shen (2020) examines the application of the Logistic sigmoid growth model of COVID-19 spread in China and its worldwide ramifications, Ademola and Sunday (2020) defined and investigated a novel continuous model known as the Gompertz Exponential distribution, the resulting densities and statistical properties were carefully calculated. Also, Jane *et al*. (2020) studied the growth curve of sugarcane varieties using non-linear models. Abd Al-Rahman *et al*. (2022) analyzed confirmed COVID-19 cases in Egypt through some new modelling sigmoidal growth curves. Al-Ghaish and Mohamed (2023) investigated the factors influencing the sustainability of Egypt's external debt. Soto *et al.* (2023) studied reflect impacts of some sigmoidal curves and dynamics of aquatic invasive species. Abul Nasr and Mahmoud (2024) studied the impact of some macroeconomic variables on Egypt's external debt over the period 1992 to 2022.

Some researchers found another technique to transform the distribution to sigmoid function, with focused computational, mathematical, modelling, and approximate challenges with the sigmoidal function and the Heaviside step function. The Hausdor approximation for the Heaviside step function using sigmoid functions is examined from a variety in computational and modelling perspectives. For example, Kyurkchiev and Markov (2015). suggested some sigmoid functions based on transmuted function transformation and using some approximation and modelling aspects. Anguelov and Markov (2016). introduced Hausdorff continuously interval formulas and approximations by Sigmoid Logistic Functions. Iliev, Kyurkchiev and Markov (2017) introduced Hausdorff approximation for the Heaviside step function by multiple sigmoid functions: log–logistic, generalized logistic, and transmuted log–logistic functions. Kyurkchiev (2018) introduced new transmuted based on Gompertz function, also Kyurkchiev (2022) introduced a remark on a Hypothesis piecewise Simplified sigmoid growth function.

Some traditional growth models can be represented mathematically as follows:

Gompertz model: $$y_i = a + (\beta - a)\, e^{-e^{(k(x_i - \gamma))}} + \varepsilon_i \,, \qquad (1)$$

Exponential model: $$y_i = a + (\beta - a)\, e^{\left(-\frac{x_i}{\mu}\right)} + \varepsilon_i \,, \qquad (2)$$

Weibull model: $y_i = a \left(1 - e^{-(k x_i)^c}\right) + \varepsilon_i$, (3)

where y_i ; $i = 1, \dots, n$ is the response variable, γ is the point of inflection, x_i is the explanatory variable, a, β, k, and c are parameters must be estimated which are defined as: a is the response variable's maximum value in the data , $a > 0$, β is the minimum of the response variable's value in the data, k is the parameter governing the rate at which the response variable approaches its potential maximum, $k > 0$, μ is the scale parameter , and c is the allometric constant, and ε_i is a random error term which assumes that it is *explanatory and identically distributed* $(i.i.d.)$ with $N(0, \sigma^2)$.

The study's purpose is to offer two new sigmoid growth models based on the Gompertz Exponential (GoE) distribution for accurately assessing diverse growth circumstances. The suggested sigmoid growth models are called the Transmuted Gompertz Exponential and Gompertz Exponential. The paper is structured as follows: Section 2 provides some approaches transformation to sigmoid functions, the new suggested sigmoidal growth models, the Gompertz Exponential and Transmuted Gompertz Exponential are introduced in Section 3. Section 4 explains the process for estimating the parameters of some single sigmoid growth models using the maximum likelihood (ML) estimation approach. The simulation research appears in Section 5. An application utilizing Egypt's external debt from 2000 to 2022 is shown in Section 6. Some closing remarks are included in Section 7.

2 Some approaches transformation to sigmoid functions

There are several approaches for transformation sigmoid functions, including: the formula based on the cumulative distribution function published by Seber and Wild (2003) and the transmuted sigmoid function introduced by Kyurkchiev and Markov (2015). To describe a sigmoid form, utilize the distribution function $F(x;\theta)$ of a random variable that is continuous having a distribution with unimodal.

2.1 The formula based on the distribution function

The distribution function $F(x;\theta)$ of exactly continuous random variable having a distribution with unimodal is used to characterise a sigmoidal shape. There are four formulas based on the cumulative distribution function, the generic equation for the sigmoid model using the distribution function is given by

$$y_1 = \beta + (a - \beta) F(k(x - \gamma); \boldsymbol{\theta}) + \varepsilon, \quad (4)$$

where y_1 is the response variable in the general formula of sigmoid model, x is the explanatory variable, γ is the point of inflection, a is the maximum value of the dependent variable in the data, $a > 0$, β is the minimum of the response variable's value in the data, k is as a scale parameter on x , $k > 0$, $\boldsymbol{\theta}$ is an unknown-parameter vector, and ε is the random error.

Also, when shifting the standard curve vertically at $\gamma = 0$ in (4), the special case of sigmoid model can be written as follows:

$$y_2 = \beta + (a - \beta) F(k x; \boldsymbol{\theta}) + \varepsilon, \tag{5}$$

where y_2 is the response variable in the special case of sigmoid model when $\gamma = 0$ and ε is the random error.

Another formula of sigmoid model as special case when $\beta = 0$ in (4) as follows:

$$y_3 = a F(k(x - \gamma); \boldsymbol{\theta}) + \varepsilon, \tag{6}$$

where y_3 is the response variable in the special case of sigmoid model when $\beta = 0$ and ε is the random error.

Also, when $\gamma = 0$ in (6), the special case of sigmoid model can be written as follows:

$$y_4 = a F(k(x); \boldsymbol{\theta}) + \varepsilon, \tag{7}$$

where y_4 is the response variable in the special case of sigmoid model when $\beta = 0, \gamma = 0$ and ε is the random error.

Some sigmoid growth functions are constructed based on distribution functions such as: The Exponential, Logistic, Log-logistic, generalized logistic, Gompertz, Weibull, generalized Weibull, Von Bertalanffy, Brody, Richards, and Monomolecular models.

2.2 Transmuted sigmoid function

An obvious way for describing a sigmoid function is to use the transmuted distribution. Transmuted distributions were introduced by Shaw and Buckley (2007). In accordance with the *quadratic rank transmutation map* (QRTM), a random variable X is considered to have a transmuted distribution. The generic equation for the sigmoid model using transmuted sigmoid function transformation is given below:

$$G(x) = (1 + \lambda) F(x) - \lambda F^2(x) \quad , \tag{8}$$

where $F(x)$: is the cdf pertaining to the base distribution, $|\lambda| \leq 1$, known as shape parameter.

Some sigmoid functions are constructed based on transmuted functions such as: the transmuted sigmoid Rayleigh function, transmuted sigmoid, log-logistic function, transmuted Gompertz function, through Hausdorff approximation.

3 The new suggested sigmoidal growth models

In this section, suggest two new sigmoid growth models using two different techniques based on the Gompertz Exponential (GoE) distribution, these models are called the Gompertz Exponential and Transmuted Gompertz Exponential. The GoE distribution was originally described in the literature by Ademola and Sunday (2020), and it is derived from the Gompertz and exponential distributions, then, Bashir and Qureshi (2022) introduced an application used the GoE distribution. The GoE distribution will be used to generate several models of sigmoid growth using cumulative distribution functions.

The cumulative distribution function of the GoE distribution can be written as follows:

$$F(x) = 1 - e^{\frac{s}{\mu}[1 - e^{\mu r x}]} , s, \mu, r > 0, \tag{9}$$

where r is scale parameter and s, μ are shape parameters.

3.1 The first suggested sigmoid growth model

From (7), the first suggested sigmoid growth model based on the distribution function is called the Gompertz Exponential (GoE) sigmoid growth model, denoted by y_i(GoE) and is written in the next form:

$$y_i(\text{GoE}) = a \left[1 - e^{\frac{s}{\mu}[1 - e^{\mu r k x_i}]} \right] + \varepsilon_i , \boldsymbol{\theta} = (a, s, \mu, k, r)^T.$$
(10)

3.2 The second suggested sigmoid growth model

From (8), the second suggested sigmoid growth model using transmuted sigmoid function is called the Transmuted Gompertz Exponential (TGoE) sigmoid growth model is denoted y_i(TGoE) and is written in the next form:

$$y_i(TGoE) =$$

$$(1 + \lambda) \left(1 - e^{\frac{s}{\mu}[1 - e^{\mu r x_i}]} \right) - \lambda (1 - e^{\frac{s}{\mu}[1 - e^{\mu r x_i}]})^2 + \varepsilon_i , \boldsymbol{\theta} = (\lambda, s, \mu, r)^T.$$
(11)

4 Estimating the parameters of some sigmoid growth models used ML technique

In this section, the parameters of the suggested model are estimated using the maximum likelihood (ML) estimation technique. The ML approach in non-linear growth models were introduced by Carolin (1990), Malott (1990). In statistics, among the most commonly used estimating techniques is the ML estimation technique, it is one of the most popular estimation techniques; the ML estimation technique estimates the parameters by solving a set of simultaneous equations. ML estimation would accomplish the estimates by using the variance and mean as parameters and identifying certain values that increase the likelihood of the observed outcomes. If the joint distribution of the ε_i in the non-linear, assuming that the model is known, the likelihood function is maximized to yield the maximum likelihood estimate of θ. Suppose the error term $\varepsilon_i's$ are $i.i.d.$ with density function $\sigma^{-1} g(\varepsilon/\sigma)$, so that g is the unit variance of the error distribution for standardized errors. Then the likelihood function is

$$f(y_I | \boldsymbol{\theta} , \sigma_\varepsilon^2) = \prod_{I=1}^{n} \left[\sigma_\varepsilon^{-1} g \left(\frac{y_I - f(x_I, \boldsymbol{\theta})}{\sigma_\varepsilon} \right) \right] = \prod_{I=1}^{n} \left[g \left(\frac{x_I - f(x_I, \boldsymbol{\theta})}{\sigma_\varepsilon^2} \right) \right], \tag{12}$$

where: $f(x_I, \boldsymbol{\theta})$ is the sigmoid growth function.

In the following sections the ML estimation technique of some sigmoid growth models will be illustrated:

4.1 Maximum likelihood estimation of the Gompertz sigmoid growth model

For the Gompertz sigmoid growth model as in (1), suppose that $\boldsymbol{y} = (y_1, \dots, y_n)^{\boldsymbol{T}}$ be n explanatory random variables with pdf, $f(y_I|\boldsymbol{\theta}, \sigma_\varepsilon^2)$ based on a parameter with a vector value $\boldsymbol{\theta}$ and the error variance, σ_ε^2. Also, the $\varepsilon_i's$ are assumed to be explanatory and $i.i.d$ with $N(0, \sigma^2)$, then the likelihood function is:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta}, \sigma_\varepsilon^2) = (2\pi\sigma_\varepsilon^2)^{-n/2} exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - \left[a + (\beta - a)\ e^{-e^{(k(x_i-\gamma))}}\right]\right)^2}{\sigma_\varepsilon^2}\right)\right]. \quad (13)$$

And the logarithm of the likelihood function is

$$l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y}) = \log(L) \propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - \left[a + (\beta - a)\ e^{-e^{(k(x_i-\gamma))}}\right]\right)^2}{\sigma_\varepsilon^2}\right). \quad (14)$$

The ML estimator $\widehat{\boldsymbol{\theta}}$ is produced by solving the next equation:

$$\left.\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}} = 0\ , \quad \boldsymbol{\theta} = (a, \beta, k, \gamma)^T\ , \quad (15)$$

where:

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial a} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left[\begin{matrix}\left[y_i - \left[a + (\beta - a)\ e^{-e^{(k(x_i-\gamma))}}\right]\right] \\ \left[1 + e^{-e^{(k(x_i-\gamma))}}\right]\end{matrix}\right], \quad (16)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \beta} = \frac{-1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{matrix}\left[y_i - \left[a + (\beta - a)\ e^{-e^{(k(x_i-\gamma))}}\right]\right] \\ \left(e^{-e^{(k(x_i-\gamma))}}\right)\end{matrix}\right), \quad (17)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial k} = \frac{(\beta - a)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{matrix}\left[y_i - \left[a + (\beta - a)\ e^{-e^{(k(x_i-\gamma))}}\right]\right] \\ \left(e^{-e^{(k(x_i-\gamma))}}\right)e^{(k(x_i-\gamma))}(x_i - \gamma)\end{matrix}\right), \quad (18)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \gamma} = \frac{-(\beta - a)\,k}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left[\begin{matrix}\left[y_i - \left[a + (\beta - a)\ e^{-e^{(k(x_i-\gamma))}}\right]\right] \\ \left(e^{-e^{(k(x_i-\gamma))}}\right)e^{(k(x_i-\gamma))}\end{matrix}\right]\right), \quad (19)$$

and

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}\left[y_i - \left[a + (\beta - a)\ e^{-e^{(k(x_i-\gamma))}}\right]\right]^2. \quad (20)$$

To obtain the ML estimators, simply sets (16) - (20) equal to zero. The resultant system of nonlinear equations can be numerically solved using the Nelder Mead maximization algorithm.

4.2 Maximum likelihood estimation of the exponential sigmoid growth model

For the exponential sigmoid growth model as in (2), suppose that $\boldsymbol{y} = (y_1, \dots, y_n)^{\boldsymbol{T}}$ be n explanatory random variables with pdf, $f(y_I|\boldsymbol{\theta}, \sigma_\varepsilon^2)$ depending on a vector valued parameter $\boldsymbol{\theta}$ and the error variance, σ_ε^2. Also, the $\varepsilon_i's$ are assumed to be explanatory and $i.i.d$ with $N(0, \sigma^2)$, then the likelihood function is:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta}, \sigma_\varepsilon^2) = (2\pi\sigma_\varepsilon^2)^{-n/2}\ exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - (a + (\beta - a)\,e^{\left(-\frac{x_i}{\mu}\right)})\right)^2}{\sigma_\varepsilon^2}\right)\right]. \tag{21}$$

The logarithm of the likelihood function expressed as $l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y})$ and is given as follows:

$$l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y}) = \log(L) \propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - (a + (\beta - a)\,e^{\left(-\frac{x_i}{\mu}\right)})\right)^2}{\sigma_\varepsilon^2}\right). \tag{22}$$

Then, the ML estimator $\widehat{\boldsymbol{\theta}}$ can be obtained by solving the next equation:

$$\left.\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}} = 0, \quad \boldsymbol{\theta} = (a, \beta, \mu)^T, \tag{23}$$

where:

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial a} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left(y_i - (a + (\beta - a)\,e^{\left(-\frac{x_i}{\mu}\right)})\right)\left[1 - e^{\left(-\frac{x_i}{\mu}\right)}\right]\right), \tag{24}$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \beta} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left(y_i - (a + (\beta - a)\,e^{\left(-\frac{x_i}{\mu}\right)})\right)e^{\left(-\frac{x_i}{\mu}\right)}\right), \tag{25}$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \mu} = \frac{(\beta - a)}{\mu^2\,\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left(y_i - (a + (\beta - a)\,e^{\left(-\frac{x_i}{\mu}\right)})\right)e^{\left(-\frac{x_i}{\mu}\right)}x_i\right), \tag{26}$$

and

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}\left(y_i - (a + (\beta - a)\,e^{\left(-\frac{x_i}{\mu}\right)})\right)^2. \tag{27}$$

To obtain the ML estimators, simply sets (24) - (27) equal to zero. The resultant system of nonlinear equations can be numerically solved using the Nelder Mead maximization algorithm.

4.3 Maximum likelihood estimation of the Transmute Gompertz Exponential sigmoid growth model

For the first new suggested model of sigmoidal growth, the Transmuted Gompertz Exponential sigmoid growth model as in (11), suppose that the $\varepsilon_i's$ are $i.i.d.N(0,\sigma^2)$, then the likelihood function becomes:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta},\sigma_\varepsilon^2) = (2\pi\sigma_\varepsilon^2)^{-\frac{n}{2}} exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i-\left[(1+\lambda)\left(1-e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right)-\lambda\left(1-e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right)^2\right]\right)^2}{\sigma_\varepsilon^2}\right)\right]. \quad (28)$$

The probability function's logarithm is represented by $l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})$ and is given as follows:

$$l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y}) = \log(L)$$

$$\propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i-\left[(1+\lambda)\left(1-e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right)-\lambda\left(1-e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right)^2\right]\right)^2}{\sigma_\varepsilon^2}\right). \quad (29)$$

Let $g_{TGoE}(x_i) = \left[(1+\lambda)\left(1-e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right) - \lambda\,(1-e^{\frac{s}{\mu}[1-e^{\mu r x_i}]})^2\right]$

Then, the ML estimator $\widehat{\boldsymbol{\theta}}$ can be acquired by the solution of the subsequent equation.:

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0\,,\ \ \boldsymbol{\theta} = (\lambda,s,\mu,r)^T\ , \quad (30)$$

where:

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\lambda} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\begin{pmatrix}(y_i-[g_{TGoE}(x_i)])\left[e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right]\\ \left[1-e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right]\end{pmatrix}, \quad (31)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial s} = \frac{-1}{\mu\sigma_\varepsilon^2}\sum_{i=1}^{n}\begin{pmatrix}(y_i-[g_{TGoE}(x_i)])\left(\left(e^{\frac{s}{\mu}[1-e^{\mu r\ i}]}\right)[1-e^{\mu r x_i}]\right)\\ \left((1+\lambda)+2\lambda[1-e^{\frac{s}{\mu}[1-e^{\mu r\ i}]}]\right)\end{pmatrix}, \quad (32)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\mu} = \frac{s}{\mu\sigma_\varepsilon^2}\sum_{i=1}^{n}\begin{pmatrix}(y_i - g_{TGoE}(x_i))\left[\left(e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right)\left(-\frac{1}{\mu} - r\,x\,e^{\mu r x_i} + \frac{e^{\mu r x_i}}{\mu}\right)\right] \\ .\left(2\,\lambda\left[1 - e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right] - (1+\lambda)\right)\end{pmatrix},$$

(33)

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial r} = \frac{s}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\begin{pmatrix}(y_i - [\,g_{TGoE}(x_i)])\left[(x_i e^{\mu r x_i})\left(e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right)\right] \\ \left[(1+\lambda) - 2\,\lambda\left(1 - e^{\frac{s}{\mu}[1-e^{\mu r x_i}]}\right)\right]\end{pmatrix}, \quad (34)$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\sigma_\varepsilon^2} = -\frac{n}{2\,\sigma_\varepsilon^2} + \frac{1}{2\,\sigma_\varepsilon^4}\sum_{i=1}^{n}(y_i - [g_{TGoE}(x)])^2. \quad (35)$$

To obtain the ML estimators, simply sets (31) - (35) equal to zero. The resultant system of nonlinear equations can be numerically solved using the Nelder Mead maximization algorithm.

4.4 Maximum likelihood estimation of the Gompertz Exponential sigmoid growth model

For the second new suggested model of sigmoidal growth, the GoE sigmoid growth model as in (10), suppose that the $\varepsilon_i's$ are $i.i.d.N(0,\sigma^2)$, then the likelihood function becomes:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta},\sigma_\varepsilon^2) = (2\,\pi\,\sigma_\varepsilon^2)^{-n/2}\,exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - a\left[1 - e^{\frac{s}{\mu}[1-e^{\mu r k x_i}]}\right]\right)^2}{\sigma_\varepsilon^2}\right)\right]. \quad (36)$$

The probability function's logarithm is represented by $l(\boldsymbol{\theta},\sigma_\varepsilon^2;\,\boldsymbol{y})$ and is given as follows:

$$l(\boldsymbol{\theta},\sigma_\varepsilon^2;\,\boldsymbol{y}) =$$

$$\log(L) \propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - a\left[1-e^{\frac{s}{\mu}[1-e^{\mu r k x_i}]}\right]\right)^2}{\sigma_\varepsilon^2}\right). \quad (37)$$

Then, the ML estimator $\widehat{\boldsymbol{\theta}}$ can be acquired by the solution of the subsequent equation:

$$\left.\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0, \quad \boldsymbol{\theta} = (a,s,\mu,k,r)^T, \quad (38)$$

where:

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial a} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{array}{c}\left(y_i - a\left[1-e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right]\right)\\ \left[1-e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right]\end{array}\right), \tag{39}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial s} = \frac{-a}{\mu\,\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{array}{c}\left(y_i - a\left[1-e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right]\right)\\ \left(e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right)\left[1-e^{\mu r k x_i}\right]\end{array}\right), \tag{40}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \mu} = \frac{a\,s}{\mu\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{array}{c}\left(y_i - a\left[1-e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right]\right)\left(e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right)\\ \left(\frac{1}{\boldsymbol{\mu}} + x_i\, r\, k\, e^{\mu r k x_i} + \frac{e^{\mu r k x_i}}{\boldsymbol{\mu}}\right)\end{array}\right), \tag{41}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial k} = \frac{a\,s\,r}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{array}{c}\left(y_i - a\left[1-e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right]\right)\\ \left(e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right)\left(e^{\mu r k x_i}\right)x_i\end{array}\right), \tag{42}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial r} = \frac{ask}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{array}{c}\left(y_i - a\left[1-e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right]\right)\\ \left(e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right)\left(e^{\mu r k x_i}\right)x_i\end{array}\right), \tag{43}$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\,\sigma_\varepsilon^2} + \frac{1}{2\,\sigma_\varepsilon^4}\sum_{i=1}^{n}\left(y_i - a\left[1-e^{\frac{s}{\mu}\left[1-e^{\mu r k x_i}\right]}\right]\right)^2. \tag{44}$$

To obtain the ML estimators, simply sets (39) - (44) equal to zero. The resultant system of nonlinear equations can be numerically solved using the Nelder Mead maximization algorithm.

5 Simulation Study

The program algorithm estimation of the ML technique for sigmoid growth models is beginning with define the sigmoid growth models formula, next compute the initial values $a_0, k_0, \beta_0, L_0, \gamma_0, r_0, s_0, \mu_0$ of the sigmoid growth models of the data and assume that $\sigma_\varepsilon^2 = 1$, then estimate parameters of the sigmoid growth models by R program package (maxLik), using Newton Raphson maximization technique to solve the derived non-linear logarithmic likelihood equations simultaneously, also evaluate the resulting estimates using the *Relative Absolute Bias* (RAB) and *Relative Mean Squared Error* (REMSE), finally, repeat the above steps N times.

- **Initial values:**

One of the most challenging issues in non-linear model parameter estimation is initial value definition. Starting values can be determined by using a preset procedure that automatically determines the first beginning values as follows:

The starting value of a: The parameter a_0 served as the upper limit of the value of the dependent variable in the data. Then new a is calculated for the different sigmoidal equations.

The starting value of k: The parameter k is the steady rate of increase in the response variable's maximum value. This definition allows one to write,

$$k = \frac{(y_n - y_1)}{a_0 (x_n - x_1)},$$

where y_1 and y_n are the response variable's values that match the initial x_1 and the last x_n observations. a_0 is the starting value specified for the parameter a.

The starting value of γ: The parameter γ is defined as the point of inflection value of the curve at explanatory variable or we can assume that γ is the value of the explanatory variable corresponding to $\frac{a_0}{2}$ value of the dependent variable.

The starting value of β: The starting value for the constant, β_0, was established by analyzing the model at the beginning of it is and assuming β as the minimum of the dependent variable in the data. Then when the predictor variable is zero, substitute with new β value for the different sigmoidal equations.

In this part, a Monte Carlo simulation can be performed to compare the performance of the suggested sigmoid growth model, Transmuted Gompertz Exponential, Gompertz Exponential, to some of the existed sigmoid growth models, such as Gompertz and Exponential. These estimators performance can be assessed using RAB and REMSE of the estimated coefficients, which are given by

$$\text{RAB}(\hat{\theta}_I) = \frac{|Mean(\hat{\theta}_I) - \theta_I|}{\theta_I}, \tag{45}$$

$$\text{REMSE}(\hat{\theta}_I) = \frac{MSE(\hat{\theta}_I)}{\theta_I}, \tag{46}$$

where $Mean(\hat{\theta}_I) = \frac{1}{N}\sum_{r=1}^{N}(\hat{\theta}_I)_r$, r is the replication of the total replications $N = 1000$, and $MSE(\hat{\theta}_I) = v(\hat{\theta}_I) + bias^2(\hat{\theta}_I)$.

5.1Simulation algorithm

The following procedures are employed to calculate the ML estimates, RAB and REMSE for the existing and suggested sigmoid growth models for varying numbers of samples $n = 20, 50, 100$ and 200. The R program (version 4.4.1) is used to build the simulation study's computation. In the subsequent phases, the performance of several sigmoid growth model estimators is compared using functions in the R program, such as minpack.lm and bbmle packages, assuming a normal distribution of random errors:

1. For the Gompertz distribution generate the explanatory variables $X_i \sim \text{gompertz}\ (1{,}1)$, for the Exponential distribution generate $X_i \sim \text{exponential}\ (1)$, and For given values of the parameters s, γ and r , the inverse cdf, can be used to generate the random variable of the explanatory (X_i) from Gompertz Exponential distribution whose cdf is given in (9), Thus, by solving the non-linear equation

   $X_i \ = \ ((log(1-((log(1-u_i))/(s/\mu))))\ /\mu\, r)\ , i = 1\ , \dots , n\ .$

   where: $u_i \sim$ standard uniform distribution (0,1).

2. Generate the values of error, ε_i from the standard normal distribution.

3. Following Caglar *et al*. (2018), can be simulated intensity noise from the uniform distribution and add the noise of parameter.

4. The initial values of the coefficients are choosing as $a \ = \ 4, s = 2, J = 2, \ \text{r} \ = \ 1.1, 0 < L < 1 \ \beta = 0.08$ for small sample sizes, and equal to 0.8 for large sample sizes, γ ranging from 0.3811 to 0.9, μ ranging from 0.9 to 1.8618 **,** and k ranging from 0.5895 to 1.3006.

5. Obtain the response variables y_i using the different equations (1), (2), (10), and (11) respectively, and add intensity noise.

6. Obtain the ML estimates by solving (15) for the Gompertz model, solving (23) for the Exponential model, solving (30) for the Transmuted Gompertz Exponential model, and solving (38) for the Gompertz Exponential model.

7. Compute the RAB and REMSE for each estimate using (45), (46) respectively.

8. Plots the fitted of the different sigmoid growth curves.

9. Repeat the above steps for all sigmoid models and all sample sizes 1000 times using R program.

   Simulation results are summarized in tables (1- 4), these tables give the estimated, RAB, and REMSE for all estimate of the single sigmoid growth models estimators.

Table 1: The average of the various sigmoid growth models predicted parameter values, RAB, RMSE for sample size 20 at a = 4, s=2, β =0.08, and r = 1.1.

Model	Estimator	Average Estimate	RAB	REMSE
Transmuted Gompertz Exponential		$\mu = 2, L = 0.1$		
	$\hat{s}$	1.85824	0.07088	0.01005
	$\hat{\mu}$	1.30950	0.34524	0.34524
	$\hat{r}$	0.66758	0.39310	0.16998
	$\hat{L}$	0.29947	1.99478	0.39791
Gompertz Exponential		$k = 1.26377, \mu = 1.8618$		
	$\hat{a}$	3.14221	0.21444	0.18394
	$\hat{s}$	1.04050	0.47974	0.46031
	$\hat{\mu}$	1.36921	0.26459	0.13034
	$\hat{k}$	1.97591	0.56350	0.40129
	$\hat{r}$	1.72727	0.57024	0.35770
Gompertz		$k = 1.30065, \gamma = 0.38111$		
	$\hat{a}$	5.35136	0.33784	0.45654
	$\hat{\beta}$	-0.12554	2.56927	0.52809
	$\hat{k}$	2.50899	0.92902	1.12257
	$\hat{\gamma}$	0.45718	0.19960	0.01518
Exponential		$\mu = 2$		
	$\hat{a}$	6.13051	0.53262	1.13477
	$\hat{\beta}$	-0.60764	8.59559	5.91073
	$\hat{\mu}$	1.66863	0.16568	0.05490

Table 2: The average of the various sigmoid growth models predicted parameter values, RAB, RMSE for sample size 50 at a = 4, s=2, β =0.8 and r = 1.1.

Model	Estimator	Average Estimate	RAB	RMSE
Transmuted Gompertz Exponential		$\mu = 2, L = 0.3$		
	$\hat{s}$	1.94168	0.02915	0.00170
	$\hat{\mu}$	2.37515	0.18757	0.07037
	$\hat{r}$	0.70132	0.36243	0.14449
	$\hat{L}$	0.58260	0.94201	0.26621
Gompertz Exponential		$k = 0.79348, \mu = 0.9$		
	$\hat{a}$	3.30073	0.17481	0.12224
	$\hat{s}$	1.26149	0.36925	0.27269
	$\hat{\mu}$	1.01285	0.12539	0.01415
	$\hat{k}$	1.12751	0.42095	0.14061
	$\hat{r}$	1.30279	0.18435	0.03738
Gompertz		$k = 0.81708, \gamma = 0.9$		
	$\hat{a}$	4.93480	0.23370	0.21846
	$\hat{\beta}$	0.85219	0.06524	0.00340
	$\hat{k}$	1.43551	0.75687	0.46807
	$\hat{\gamma}$	1.05639	0.17376	0.02717
Exponential		$\mu = 2$		
	$\hat{a}$	5.24063	0.31015	0.38479
	$\hat{\beta}$	-0.24797	1.30996	1.37281
	$\hat{\mu}$	3.75835	0.87917	1.54590

Table 3: The average of the various sigmoid growth models predicted parameter values, RAB, RMSE for sample size 100 at a = 4, s=2, β =0.8 and r = 1.1.

Model	Estimator	Average Estimate	RAB	RMSE
	$\mu = 2, L = 0.7$			
Transmuted Gompertz Exponential	$\hat{s}$	2.00630	0.00315	0.00001
	$\hat{\mu}$	1.95040	0.02479	0.00122
	$\hat{r}$	1.22220	0.11109	0.01357
	$\hat{L}$	0.70285	0.00408	0.00001
	$k = 0.75448, \mu$ =0.9			
Gompertz Exponential	$\hat{a}$	4.54812	0.13703	0.07510
	$\hat{s}$	1.84613	0.07693	0.01183
	$\hat{\mu}$	0.94257	0.04730	0.00201
	$\hat{k}$	0.73092	0.03123	0.00073
	$\hat{r}$	1.23898	0.12634	0.01755
	$k = 0.58957, \gamma$ =0.9			
Gompertz	$\hat{a}$	4.09552	0.02388	0.00228
	$\hat{\beta}$	0.81036	0.01295	0.00013
	$\hat{k}$	0.93651	0.58844	0.20415
	$\hat{\gamma}$	0.92168	0.02409	0.00052
	μ=2			
Exponential	$\hat{a}$	4.33829	0.08457	0.02861
	$\hat{\beta}$	0.54832	0.31458	0.07917
	$\hat{\mu}$	1.70818	0.14590	0.04257

Table 4: The average of the estimated parameter values of the different growth models, RAB, RMSE for sample size 200 at a = 4, s=2, β =0.8 and r = 1.1.

Model	Estimator	Average Estimate	RAB	RMSE
	$\mu = 2, L = 0.99$			
Transmuted Gompertz Exponential	$\hat{s}$	2.00287	0.00143	0.00000
	$\hat{\mu}$	1.99887	0.00056	0.00000
	$\hat{r}$	0.99756	0.09312	0.00953
	$\hat{L}$	0.99015	0.00015	0.00000
	$k = 0.76301, \mu = 0.9$			
Gompertz Exponential	$\hat{a}$	4.01869	0.00467	0.00008
	$\hat{s}$	2.06384	0.03192	0.00203
	$\hat{\mu}$	0.89395	0.00671	0.00004
	$\hat{k}$	0.77947	0.02156	0.00035
	$\hat{r}$	1.12528	0.02298	0.00058
	$k = 0.77544, \gamma$ =0.9			
Gompertz	$\hat{a}$	4.06966	0.01741	0.00121
	$\hat{\beta}$	0.80728	0.00910	0.00006
	$\hat{k}$	0.82715	0.06668	0.00347
	$\hat{\gamma}$	0.90264	0.00294	0.00001
	μ =2			
Exponential	$\hat{a}$	4.20625	0.05156	0.01064
	$\hat{\beta}$	0.59158	0.26051	0.05423
	$\hat{\mu}$	1.85276	0.07361	0.01083

The fitted growth curves are illustrated using R program and are shown in Figures (1 – 4) as follows:

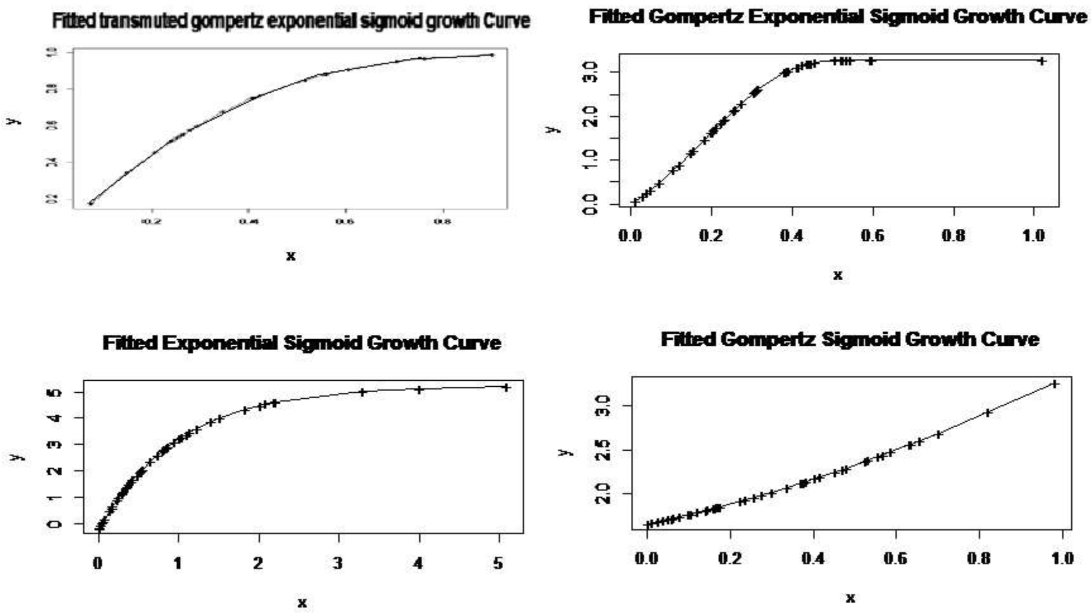

Figure 1 : Plots of the fitted sigmoid growth curves when $n = 20$.

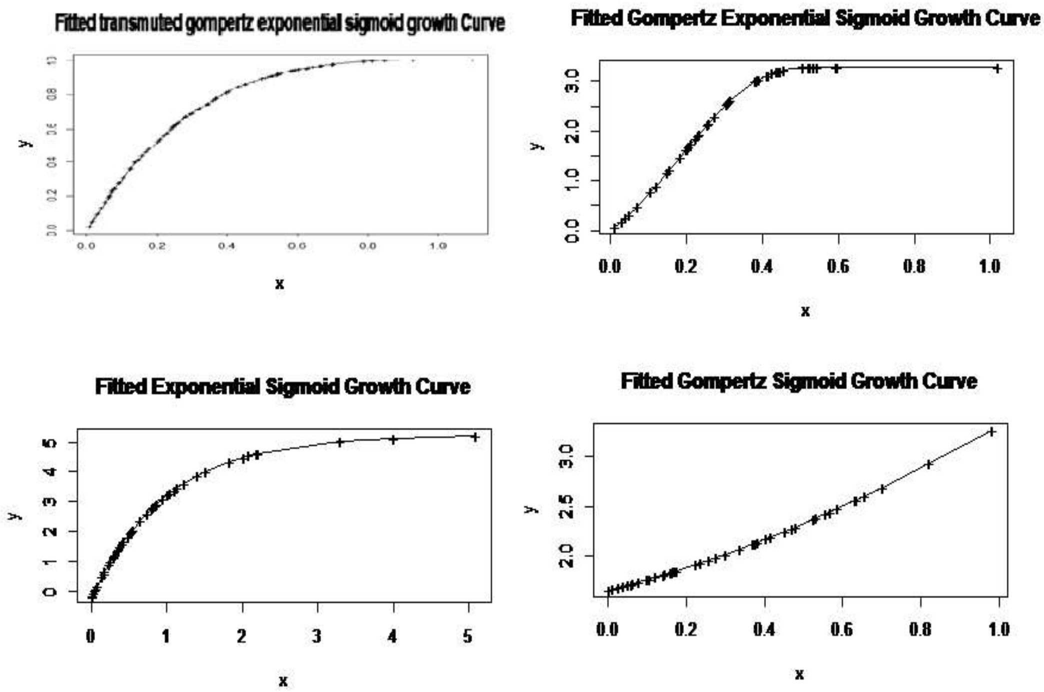

Figure 2 : Plots of the fitted sigmoid growth curves when $n = 50$.

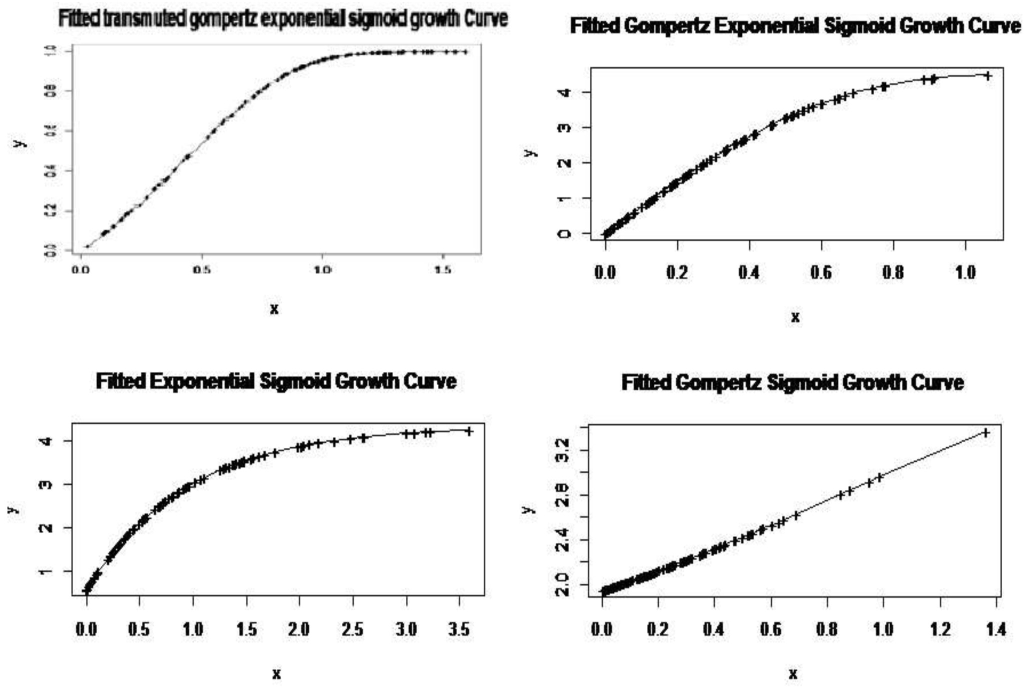

Figure 3 : Plots of the fitted sigmoid growth curves when $n = 100$.

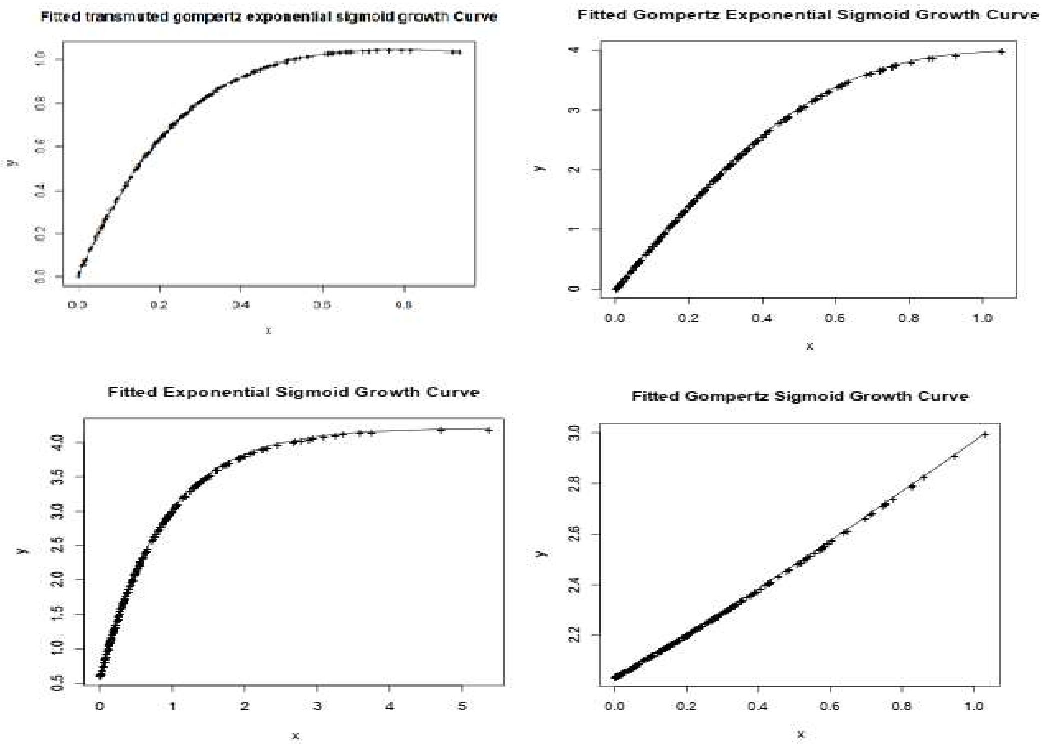

Figure 4 : Plots of the fitted sigmoid growth curves when $n = 200$.

4.2 Simulation results

The simulation study's primary findings are as follows:

- According to the theoretical conclusions, the RAB and REMSE dropped as n increased.
- As n increases, the average estimate values were very close to almost all initial sample sizes in all models.
- It is found that, as k value decreased, and the RAB, REMSE decreased, for different sigmoid models in most sample sizes.
- As L increases, the average estimate values were very close to almost all initial sample sizes, and the RAB, REMSE decreased in all models.
- As shown in Figures (1 – 4), there are no outliers, and the curves have the sigmoidal shape " S-shaped " curve.

6 Application

The issue of Egyptian external debt has become noticeable as the amount of this debt has grown significantly, hitting record levels in the last ten years. This growth is important for implementing needed structural changes and boosting the Egyptian economy. Effectively handling the responsibilities of paying off external debt on time is vital to ensure it does not hinder the pursuit of desired growth rates. The research was based on World Bank data for the period from 2000 to 2022, using the R program, and Transmuted Gompertz Exponential, Gompertz Exponential, Gompertz, and Exponential sigmoid growth models.

To check the performance of the new suggested sigmoid growth models, the data set on the value of Egyptian external debt yearly from 2000 to 2022. The explanatory variable considered in this study is the number of years (x) and the value of Egyptian external debt in million dollars (y) is considered as a response variable, the data was recorded every year.

29.2, 28.3, 28.6, 30.4, 31.4, 30.5, 29.5, 34.6, 33.9, 35.4, 33.6, 35, 36, 43.3, 46.1, 48.6, 55.8, 82.8, 96.6, 108.8, 123.5, 137.8, 157.7 .

Figure 5 displays the relationship between the value of Egyptian external debt as response variable (y), and the years as explanatory variable (x) after the data are refined by applying the transformation of inverted variance.

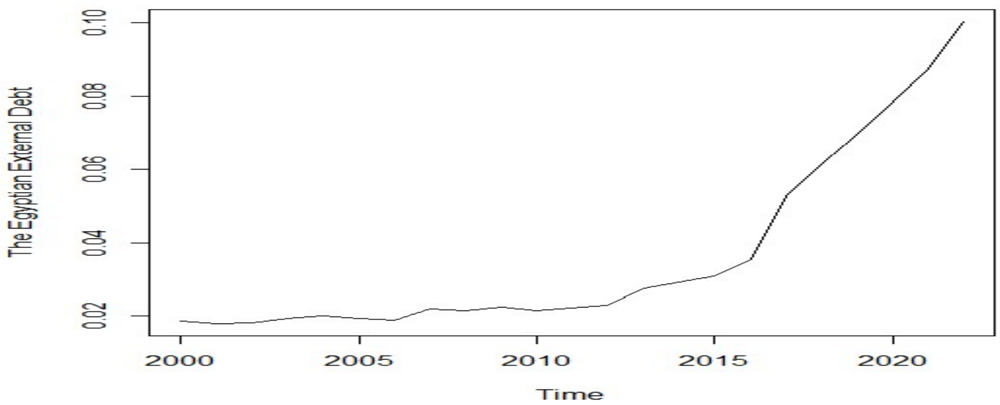

Figure. 5 : Description of the value of Egyptian external debt over time.

The initial values are calculated as $a_0 = 0.100317$, $\beta_0 = 0.018002, k_0 = 0.03729, \gamma_0 = 0.03$, $\mu_0 = 0.056$, $s_0 = 0.0005, 0 < L_0 < 1$ and $r_0 = 1.7$. Plots of growth curves, Transmuted Gompertz Exponential, Gompertz Exponential, Gompertz and Exponential are displayed in Fig. 2. Also, fitted growth curves of the Transmuted Gompertz Exponential, Gompertz Exponential, Gompertz and Exponential growth models for the data set are displayed in Fig. 3. Estimation of the model parameters are performed by ML technique are obtained by Newton-Raphson maximization using maxLik package of (R.4.4.1) Table 5 shows the parameter estimates by ML estimation and *Approximate Standard Error* (ASE). Also, for comparison between the models, the *Akaike Information Corrected criterion* (AICc) and *Likelihood Ratio Test* (LRT) are used in (Table 6).

The *Akaike Information Corrected criterion* (AICc) is a formula that adds a correction term to the *Akaike Information criterion* (AIC) to give a more accurate answer for smaller samples. AICc is the sum of AIC and an additional non-stochastic penalty term. The AICc is computed as follows:

$$\text{AICc} = -2\, l + 2\, b + \frac{2\, b\, (b+1)}{n-b-1}, \tag{47}$$

where l is the logarithm of likelihood function for the model, and b represents the number of the model's parameter count.

The quality of fit of two statistical models is compared using the *likelihood ratio tests* (LRT). One way to compare nested models is via a Likelihood Ratio test. When two models are " nested ", it indicates that one is a special case of the other, with fewer parameters fitted. Many refer to these as the " full " (more complicated) and " reduced " (simpler) models. In order to compare nested models, the LRT statistic is calculated as follows when the maximum likelihood approach is applied to fit the data:

$$\text{LRT} = 2\, log\left(\frac{L_{full}}{L_{reduced}}\right) = 2\, \left(log\left(L_{full}\right) - log\left(L_{reduced}\right)\right), \tag{48}$$

where L_{full} and $L_{reduced}$ are, respectively, the likelihood functions for the complete and reduced models.

This function is intimately associated with the residual sum of squares. It is assumed that LRT is approximately χ^2 distributed with r degrees of freedom, where r is the difference in the number of fitted parameters between the complete and reduced models

For evaluating the selection models to the data, the following criteria are used: the coefficient of determination, R^2, *Mean Squared Error* (MSE) and *Root Mean Squared Error* (RMSE) as shown in Table 7 according to the following formulas:

$$\mathrm{R}^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2} , \tag{49}$$

$$\mathrm{MSE} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n - b - 1} , \tag{50}$$

$$\mathrm{RMSE} = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n - b - 1}} , \tag{51}$$

where n is the sample size, y_i , $\hat{y}_i$ are the actual and anticipated values, respectively, $\bar{y}$ is the mean of observed values, and b is the number of parameters in the model.

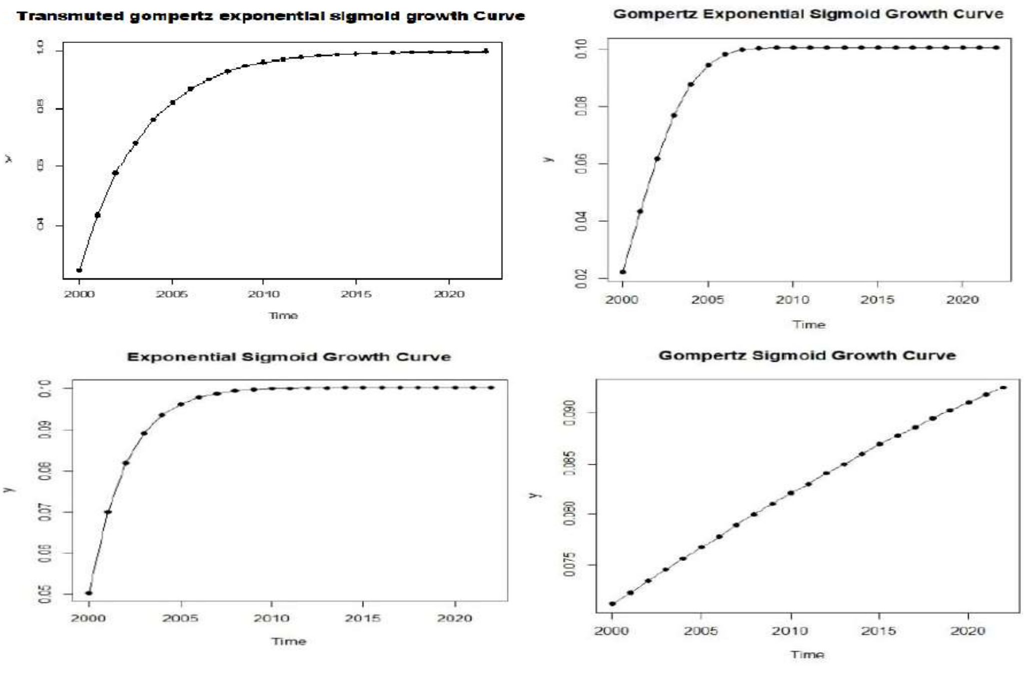

Figure. 6 : Plots of sigmoid growth curves.

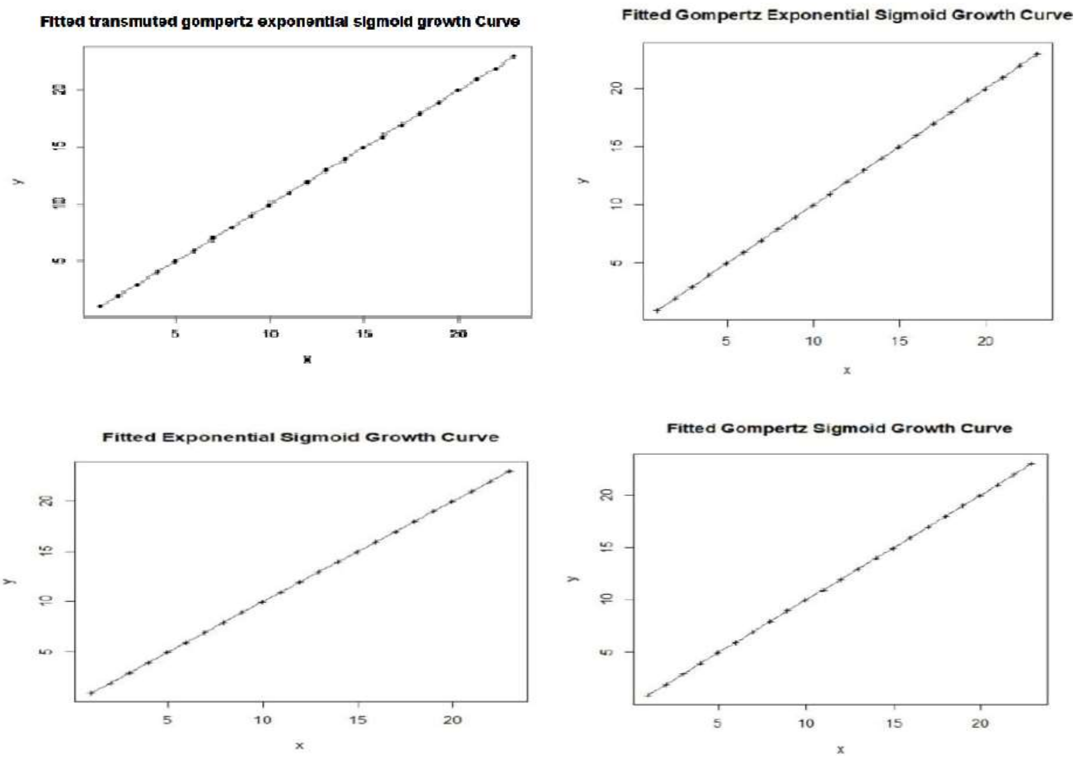

Figure. 7 : Plots of the fitted sigmoid growth curves.

Table 5. Parameter estimates, approximate and standard errors of parameters for Transmuted Gompertz Exponential, Gompertz Exponential, Gompertz, and Exponential sigmoid growth model.

Model	**parameter**	**Estimate**	**ASE**
	s	0.00193	0.03051
Transmuted Gompertz Exponential	μ	0.01706	15.85994
	r	0.02547	2.02374
	L	0.91623	0.01024
	a	0.02011	0.31023
	s	0.00217	0.14860
Gompertz Exponential	μ	0.48619	24.01011
	k	0.13456	6.02675
	r	1.57427	1.03541
	a	0.10351	2.20913
Gompertz	β	0.01896	0.33715
	k	0.37456	13.06731
	v	10.30785	52.74021
	a	0.47421	4.26652
Exponential	β	-0.00193	0.44091
	μ	0.06218	20.53771

Table 6. Evaluation of AICc, and *p-values* of LRT test for Transmuted Gompertz Exponential, Gompertz Exponential, Gompertz and Exponential sigmoid growth models

Model	AICc	*p-value*
Transmuted Gompertz Exponential	7.830	2.2×10^{-16}
Gompertz Exponential	8.053	2.2×10^{-16}
Gompertz	10.222	2.2×10^{-1}
Exponential	17.267	2.2×10^{-16}

Table 7. The R^2, MSE, and RMSE for Transmuted Gompertz Exponential, Gompertz Exponential Gompertz and Exponential sigmoid growth models

Model	R^2	MSE	RMSE
Transmuted Gompertz Exponential	0.99921	0.00010	0.01000
Gompertz Exponential	0.99846	0.00011	0.01049
Gompertz	0.99263	0.00025	0.01581
Exponential	0.98720	0.00037	0.01924

The tables and fingers show that the LRT is significant ($p - value < 0.05$) in all models, and all evaluated models fitted well the investigated curves of Egyptian external debt with R^2, MSE and RMSE values. The Transmuted Gompertz Exponential, and Gompertz Exponential models, are best suited to characterize Egyptian foreign debt accumulation over time due to its low AICc, the best since it had the largest value of R^2 and the lowest value of MSE and RMSE.

6. Conclusions

In this research, suggested new sigmoid growth models are given that can capture the most diverse growth data scenarios. The new suggested models are based on the Gompertz Exponential distribution and employ the cdf formula using two different techniques, these models called the Gompertz Exponential, and Transmuted Gompertz Exponential sigmoid growth models. The ML estimation technique was used to determine the parameters of the new suggested models and some existing sigmoid growth models. Furthermore, the performance of the suggested and existed sigmoid growth models was examined using a Montecarlo simulation and, also application on the Egyptian foreign debt data from 2000 to 2022. The results show that Transmuted Gompertz Exponential sigmoidal growth model outperformed the Gompertz Exponential sigmoidal growth model, and other models in terms of R^2, MSE, RMSE, and AICc.

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