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Robust Estimation for Beta Regression Model in the Presence of Outliers: A Comparative Study

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Abstract

Beta regression model is a widely known statistical model when the response variable has the form of fractions or percentages. The maximum likelihood method is usually employed for estimating the regression coefficients of beta regression model. However, the maximum likelihood estimator is highly sensitive to outliers. To solve this problem, new statistical techniques have been developed that are not so easily affected by outliers; these are robust methods. This paper discusses the efficiency of using four robust estimation methods, the S-estimation method, MM-estimation method, least trimmed sum of absolute deviation method and the robust and efficient weighted least squares estimation method in estimating the parameters of beta regression model in the presence of outliers. A Monte Carlo simulation study is performed to compare the performance of these robust estimators with the maximum likelihood estimator. Also, a real data set is used to illustrate the applicability of these estimators. The results showed that the robust and efficient weighted least squares estimation method gives better performance than the maximum likelihood estimation, S-estimation, MM-estimation, and least trimmed sum of absolute deviation methods.

Keywords: *Beta regression model, Outliers, S-estimation, MM-estimation, Robust and efficient weighted least squares estimation.*

1. Introduction

The beta regression model is a widely known statistical model when the dependent variable has the form of fractions or percentages and is commonly used in many areas, such as medicine, environment, finance, and natural sciences. The *maximum likelihood estimation* (MLE) is widely used to make inferences for the parameters for beta regression model. Vasconcellos and Cribari-Neto (2005) studied the behavior of MLE in beta regression model where the parameters of the distribution are nonlinear functions of linear combinations of the explanatory variables with unknown coefficients. Also, Ospina *et al.* (2006) introduced MLE to estimate the unknown parameter and considered the bias correction mechanism based on the parametric bootstrap for beta regression model.

It is well-known that the maximum likelihood estimator-based inference suffers from the lack of robustness in the presence of outliers. Such a case can bring severe bias and misleading conclusions [Maluf *et al.* (2022)].

Rousseeuw and Leroy (2003) described two types of outliers data in the regression. First, the observed data shows a large difference between the regression vectors and the central point of the data; such case is called the leverage point. The second, the observed data shows a large difference between the response variable and the mean of predicted variable by the model is called a regression outlier. To remedy this problem, new statistical techniques have been developed that are not so easily affected by outliers. These are robust methods, such as M-estimation, *least median of squares* (LMS), *least trimmed squares* (LTS), S-estimation, MM-estimation, *least trimmed sum of absolute deviation* (LTA), and *robust and efficient weighted least squares estimation* (REWLSE).

There are several studies that deal with robust estimation methods in the regression models, such as Alma (2011) presented four robust regression methods M-estimation, MM-estimation, S-estimation, and

LTS estimation and compared these methods with the *ordinary least squares* (OLS) regression method when data contains outliers. Çankaya and Abacı (2015) compared some robust estimation methods: M-estimation, MM-estimation, S-estimation, and LTS with OLS in the presence of outliers in linear regression model. Yu and Yao (2017) studied and discussed some robust methods for linear regression models, such as M-estimation, MM-estimation, LMS, LTS, S-estimation, generalized M-estimation, R-estimation, *least absolute deviation* (LAD), and REWLSE. Almetwally and Almongy (2018) compared the M-estimation, S-estimation, and MM-estimation methods to determine the better method for a linear regression model. Tinungki (2018) used the MM-estimation method based on Tukey bisquare objective function to overcome the presence of severe outlier data. Abd El-Raoof *et al.* (2022) introduced a comparison between the performance of robust M-estimation and MM-estimation methods with the maximum likelihood method for estimating the parameters of Weibull regression model in the presence of outliers. Singgih and Fauzan (2022) presented a comparison of the M-estimation, S-estimation, and MM-estimation methods to find the optimum estimation of robust regression in criminal cases in Indonesia. A few studies for robust estimators in beta regression model have been provided, for example Ghosh (2019) proposed the robust minimum density power divergence estimator and a class of robust Wald-type tests for the beta regression model. Maluf *et al.* (2022) presented the maximum likelihood estimation method and robust estimators through the logit transformation for the beta regression model.

The main goal of this paper is to compare the performance of the MLE method, and four robust estimation methods, the S-estimation, MM-estimation, LTA estimation, and REWLSE method for estimating the parameters of beta regression model in the presence of outliers. Section 2 presents a review on beta regression model. Section 3 contains the MLE method to estimate the parameters of the beta regression model.

Section 4 suggests the S-estimation, MM-estimation, LTA estimation, and REWLSE methods for estimating the parameters of the beta regression model. Section 5 investigates the effectiveness of the MLE, S-estimation, MM-estimation, LTA estimation and REWLSE methods in the beta regression model in the presence of outliers using a Monte Carlo simulation study. Section 6 presents an application of the tropical tuna percentage real data set. Finally, the conclusions are presented in Section 7.

2. Beta Regression Model

The beta regression model was first developed by Ferrari and Cribari-Neto (2004) by connecting the mean function of the response variable to a set of linear predictors via a monotone differentiable function called the link function.

Let y be a continuous random variable having a beta distribution with probability density function of y is given as follows:

$$f(y;p,q)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,y^{p-1}(1-y)^{q-1}\,,\ 0<y<1, p>0, q>0, \quad (1)$$

where $\Gamma(.)$ is the gamma function, p and q are two shape parameters.

The mean and the variance of y are as follows:

$$E(Y)=\frac{p}{p+q}, \quad (2)$$

$$Var(Y)=\frac{pq}{(p+q)^2(p+q+1)}. \quad (3)$$

To obtain a regression structure for the mean of the response along with a precision parameter, a different parameterization of the beta density will be obtained.

Let $\mu=\frac{p}{p+q}$ and $\varphi=p+q$, $p=\mu\varphi$, and $q=(1-\mu)\varphi$. Then, the density of y can be written in the new parameterization as follows:

$$f(y;\mu,\varphi)=\frac{\Gamma(\varphi)}{\Gamma(\mu\varphi)\Gamma((1-\mu)\varphi)}\,y^{\mu\varphi-1}(1-y)^{(1-\mu)\varphi-1}\,,\ 0<y<1,\ 0<\mu<1,\ \varphi>0, \quad (4)$$

where μ is the mean of the response variable and φ is the precision parameter. Consequently, the mean and variance based on the new parameterization are as follows:

$$E(Y) = \mu, \quad (5)$$

$$Var(Y) = \frac{\mu(1-\mu)}{1+\varphi}. \quad (6)$$

The variance of the response variable decreases as the φ is increased for fixed μ. If $\mu\varphi > 1$ and $(1-\mu)\varphi > 1$, the beta density is bounded, has a single mode in (0,1), and decreases to zero. While, $\mu\varphi < 1$ and $(1-\mu)\varphi < 1$, the beta density is unbounded at one or both boundaries.

The beta regression model was developed assuming a homogeneous precision parameter in the form of generalized linear model for the location parameter using a link function.

Let $y_1, y_2, \dots, y_n$ be independent random variables, where each y_t, $t = 1,2,\dots,n$ follows the density as given in Equation (4) with mean μ_t and unknown precision φ. The beta regression model is obtained by assuming that $y_t \sim \beta eta(\mu_t, \varphi), t = 1, 2, \dots, n$ and the logit link function is defined as:

$$g(\mu_t) = log\left(\frac{\mu_t}{1-\mu_t}\right) = \eta_t = \sum_{t=1}^{n} X'_{tj}\,\beta_j, j = 1,2,\dots,k, \quad (7)$$

where g(·) is the link function of the beta regression model used to relate the systematic component with the random component,

$\beta = (\beta_0, \beta_1, \beta_2, \dots, \beta_k)'$ is a $(k+1) \times 1$ vector of unknown regression coefficients, $X_t = (1, x_{t1}, x_{t2}, \dots, x_{tk})$ is t th row of the $n \times (k+1)$ data matrix with k explanatory variables, and η_t is the linear predictor.

There are several possible choices for the link function can be used, such as the logit link function $g(\mu_t) = log\,\{-log(1-\mu)\}$, the probit function $g(\mu) = \phi^{-1}(\mu)$, where $\phi(.)$ is the cumulative distribution function of the standard normal distribution function.

A particularly interesting specification of the model is obtained when the logit link function is used. In this case, the mean of y_t can be written as Ferrari and Cribari-Neto (2004) by

$$\mu_t = \frac{e^{X_t'\beta}}{1+e^{X_t'\beta}}, \quad (8)$$

where μ_t is the mean response function. Since η depends on β and the mean response μ is a function of η, the means $\mu_1, \mu_2, \dots, \mu_n$ are functions of β [Abonazel *et al.* (2022)].

3. Maximum Likelihood Estimation

Estimation of the beta regression parameters is done by using the MLE method by Ospina *et al.* (2006). The log-likelihood function for the beta regression model has the form

$$l(\beta,\varphi) = \sum_{t=1}^{n} l_t(\mu_t,\varphi) = \sum_{t=1}^{n}\{log\Gamma(\varphi) - log\Gamma(\mu_t\varphi) - log\Gamma(\varphi - \mu_t\varphi) + (\mu_t\varphi - 1)\,log(y_t) + (\varphi - \mu_t\varphi - 1)log\,(1-y_t)\}. \quad (9)$$

When the MLE is used to estimate the unknown regression coefficients, the score function can be obtained by differentiating the log-likelihood function with respect to β and φ as follows:

$$S(\beta) = \frac{\partial l(\beta,\varphi)}{\partial \beta_j} = \sum_{t=1}^{n} \frac{\partial l(\mu_t,\varphi)}{\partial \mu_t}\frac{d\mu_t}{d\eta_t}\frac{\partial \eta_t}{\partial \beta_j} = \sum_{t=1}^{n} \frac{(y_t-\mu_t)}{\varphi Var(\mu_t)}\frac{d\mu_t}{d\eta_t} x_{tj}, \quad (10)$$

$$S(\varphi) = \frac{\partial l(\beta,\varphi)}{\partial \varphi} = \sum_{t=1}^{n}\{\mu_t(y_t^* - \mu_t^*) + log(1-y_t) + \psi(\varphi) - \psi((1-\mu_t)\varphi)\}, \quad (11)$$

where $Var(\mu_t) = \mu_t(1-\mu_t)$, x_{tj} is the $t\ th$ values for $j\ th$ covariates, $y_t^* = log\left(\frac{y_t}{1-y_t}\right)$, $\mu_t^* = \psi(\mu_t\varphi) - \psi\big((1-\mu_t)\varphi\big)$, and $\psi(.)$ denotes the digamma function.

The MLE of β and φ are obtained as the solution of the nonlinear system. In practice, the maximum likelihood estimates can be obtained through numerical maximization of (9) using a nonlinear optimization algorithm.

Since the Equation (10) is nonlinear in β, the solution of $S(\beta)$ is equating to be zero, which can be found through Fisher's scoring iterative procedure within the Newton Raphson methods for estimating parameter vector β as:

$$\beta_{m+1} = \beta_m + \left[\frac{\partial^2 l(\beta)}{\partial \beta_j \beta_k}(\beta_m)\right]^{-1} S(\beta_m), \tag{12}$$

where $S(\beta_m)$ is the score function and $m = 0, 1, 2, \dots, M$ are the iterations. The second derivative of Equation (9) with respect to β_j is as follows:

$$\frac{\partial^2 l(\beta,\varphi)}{\partial \beta_j \beta_k} = \sum_{t=1}^{n} \varphi^{-1} \left(\frac{\partial l(\beta,\varphi)}{\partial \beta_j}\right) \frac{(y_t - \mu_t)}{Var(\mu_t)} \frac{d\mu_t}{d\eta_t} x_{tj}$$

$$= \sum_{t=1}^{n} \varphi^{-1} \left[\left\{Var(\mu_t)^{-1} \left(\frac{d\mu_t}{d\eta_t}\right)^2\right\} - (\mu_t - y_t) \times \left\{Var(\mu_t)^{-2} \left(\frac{d\mu_t}{d\eta_t}\right)^2 \frac{\partial Var(\mu_t)}{\partial \mu_t} - Var(\mu_t)^{-1} \left(\frac{\partial^2 Var(\mu_t)}{\partial \eta_t^2}\right)\right\}\right] x_{tj} x_{tk}, \tag{13}$$

where $S(\beta_m)$ and $\frac{\partial^2 l(\beta)}{\partial \beta_j \beta_k}(\beta_m)$ are computed at β_m.

The initial value for each precision parameter is $\hat{\varphi}_t = \left(\frac{\hat{\mu}_t(1-\hat{\mu}_t)}{\hat{\sigma}_t^2}\right)$, where $\hat{\mu}_t$, $\hat{\sigma}_t^2$ values are obtained from linear regression. Given $m = 0,1,2,\dots,M$ is the number of iterations that are performed, the maximum likelihood estimator of β_m is obtained as *iterative reweighted least squares* (IRLS).

$$\hat{\beta}_m = \left(X'\widehat{W}_m X\right)^{-1} X'\widehat{W}_m \hat{z}_m, \tag{14}$$

where $\widehat{W}_m = \text{diag}\left[\left\{g(\hat{\mu}_t)\right\}^2 Var(\hat{\mu}_t)\varphi\right]^{-1}$, $g(.)$ is the logit link function, and $\hat{z}_m = log(\hat{\mu}_t) + \frac{y_t - \hat{\mu}_t}{\hat{\mu}_t(1-\hat{\mu}_t)}$.

However, $\hat{\beta}_m$ converges to $\hat{\beta}_{MLE}$ as $m \to \infty$ was concluded, then, the maximum likelihood estimator of β is obtained as:

$$\hat{\beta}_{MLE} = \left(X'\widehat{W}X\right)^{-1} X'\widehat{W}\hat{z}_m. \quad (15)$$

[Qasim *et al.* (2021) and Abonazel *et al.* (2022)]

4. Robust Estimation Methods

Robust regression is an important method for analyzing data that are contaminated with outliers. It can be used to detect outliers and to provide resistant results in the presence of outliers. In order to achieve this stability, four robust estimation methods in the following sub sections are suggested: the S-estimation, MM-estimation, LTA estimation, and REWLSE methods to estimate the parameters of the beta regression model in the presence of outliers.

4.1. S- estimation

Rousseeuw and Yohai (1984) proposed S-estimation to overcome the low breakdown point of M-estimation by minimizing a robust M-estimate of the residual scale. To estimate the parameters of the beta regression model using the S-estimation method, it is required to

$$min \sum_{t=1}^{n} \rho\left(\frac{r_t}{\hat{\sigma}_s}\right), \quad (16)$$

where ρ function possesses the following properties $\rho(r_t) \geq 0$, $\rho(0) = 0$, and $\rho(r_t) = \rho(-r_t)$, $r_t = y_t - \hat{y}_t$, and $\hat{\sigma}_s$ is the robust scale estimator given by

$$\hat{\sigma}_s = \begin{cases} \dfrac{median|r_t - median(r_t)|}{0.6745} & ; iteration = 1 \\ \sqrt{\dfrac{1}{nK}\sum_{t=1}^{n} w_t r_t^2} & ; iteration > 1, K = 0.199, \end{cases} \quad (17)$$

and the weighted function w_t is defined by $w_t = w(u_t) = \left(\frac{\rho(u_t)}{u_t^2}\right)$,

which is as follows:

$$w(u_t) = \begin{cases} \left[1 - \left(\frac{u_t}{c}\right)^2\right]^2 & , |u_t| \le c \\ 0 & , |u_t| > c , \end{cases} \tag{18}$$

where $u_t = \left(\frac{r_t}{\hat{\sigma}_s}\right)$ and c =1.547 is the tuning parameter.

To solve Equation (16), set the first partial derivatives of it with respect to the regression coefficients equal to zero, then,

$$\sum_{t=1}^{n} x_{tj} \rho'(\frac{r_t}{\hat{\sigma}_s}) = 0, j = 0,1, \dots, k, \tag{19}$$

where $\rho'(.)$is the first partial derivative of the function ρ, known as the influence function.

For the ρ function, the Tukey's bisquare objective function by Beaton and Tukey (1974) can be used as

$$\rho(u_t) = \begin{cases} \frac{u_t^2}{2} - \frac{u_t^4}{2c^2} + \frac{u_t^6}{6c^4} , & |u_t| \le c \\ \frac{c^2}{6} , & |u_t| > c , \end{cases} \tag{20}$$

where c is the tuning constant chosen as $c = 4.685$.

The solution of (16) is obtained by using the IRLS.

4.2. MM- estimation

The MM-estimation method is a modified type of M-estimation method that was developed by Yohai (1987) by combining a method of high breakdown value and an efficient estimation method to achieve the breakdown point up to 50% and asymptotic efficiency as close to one as desired. The MM-estimator obtains the scaled errors by the S-estimator.

The following steps are performed to estimate the parameters of beta regression model.

- Estimate regression coefficients on the data using the OLS.
- Calculate the residual value $r_t = (y_t - \hat{y}_t)$.
- Calculate the value of $\hat{\sigma}_s$ using the S-estimator, then, $u_t = \frac{r_t}{\hat{\sigma}_s}$.
- Calculate the weighted value $w_t = \left(\frac{\rho'(u_t)}{u_t}\right)$, $t = 1,2,\dots,n$.
- Calculate $\hat{\beta}_{\text{MM}}$ using the IRLS method with the weighted w_t as follows:

$$\hat{\beta}_{\text{MM}} = \left(X'\widehat{W}_{\text{MM}}X\right)^{-1}X'\widehat{W}_{\text{MM}}Y, \qquad (21)$$

where $\widehat{W}_{\text{MM}} = diag(w_1, w_2, \dots, w_n)$ and

$$w_t = \begin{cases} [1 - (\frac{u_t}{4.685})^2]^2, & |u_t| \le 4.685 \\ 0, & |u_t| > 4.685 \end{cases}.$$

4.3. Least trimmed sum of absolute deviation

The LTA robust estimator for univariate data was first developed by Bassett (1991) and Tableman (1994 a, b), and it was further demonstrated to have desirable asymptotic qualities such robustness, consistency, high breakdown, and normality. They provided the algorithm and the derivation of the influence function and asymptotic. They also provided a technique for quickly computing the LTA.

In the regression model, the LTA is a special case of the R-estimators. In particular for big data sets, Hawkins and Olive (1999) demonstrated that LTA is a competing alternative to LMS and LTS [Helmy *et al.* (2021)].

To estimate the parameters of the beta regression model using LTA method, through minimizing the sum of the absolute residuals

$$min \sum_{t=1}^{h} |r_t|_{t:n}, \tag{22}$$

where $|r_t|_{[t:n]} = |y_t - \hat{y}_t|_{[t:n]}$, $|r_t|_{[1:n]} \leq |r_t|_{[2:n]} \leq \cdots \leq |r_t|_{[n:n]}$ are the ordered absolute residuals, and $h = \frac{n+1}{2}, \frac{n}{2} + 1$ for odd n and even n respectively.

4.4. Robust and efficient weighted least squares estimation

Gervini and Yohai (2002) proposed a new class of robust regression method called the REWLSE. This method is much more attractive than many other robust estimators that attain high breakdown point and full efficiency under normal errors. This new estimator, *robust and efficient weighted least squares* (REWLS) is a type of weighted least squares estimator with the weights adaptively calculated from an initial robust estimator.

The following steps are performed to estimate the parameters of beta regression model:

- Consider a pair of initial robust estimates of regression parameters and scale, $\hat{\beta}_0$ and $\hat{\sigma}$ respectively.
- Find the standardized residuals $r_t = (y_t - \hat{y}_t)/\hat{\sigma}$.
- Define a measure of proportion of outliers in the sample

$$d_n = \max_{t>t_0} \left\{ F^+\left(|r|_{(t)}\right) - \frac{(t-1)}{n} \right\}^+ , \tag{23}$$

where $\{.\}^+$ denotes positive part, F^+ denotes the distribution of $|X|$ when $X \sim F$, $|r|_{(1)} \leq \ldots \leq |r|_{(n)}$, are the order statistics of the standardized absolute residuals, and $t_0 = max\{t: |r|_{(t)} < \eta\}$, where η is some large quantile of F^+. According to Rousseeuw and Leroy (1987), η is chosen equals 2.5.

- Calculate the weights as proposed by Gervini and Yohai (2002) by

$$W_i = \begin{cases} 1 & if\ |r_t| < i_n \\ 0 & if\ |r_t| \geq i_n, \end{cases} \tag{24}$$

where $i_n = |r|_{(t_n)}$ is the adaptive cut of value with $t_n = n - \lfloor nd_n \rfloor$, and $\lfloor nd_n \rfloor$ is the largest integer less than or equal to nd_n.

- The REWLS estimator of β is given by

$$\hat{\beta}_{REWLSE} = \left(X'\widehat{W}_{REWLSE}X\right)^{-1} X'\widehat{W}_{REWLSE}Y, \tag{25}$$

where $\widehat{W}_{REWLSE} = diag(W_1, W_2, \ldots, W_n)$.

5. Simulation study

In this section, a Monte Carlo simulation study is conducted to make a comparison between the performances of four robust estimation methods: the S, MM, LTA, and REWLSE with the MLE method. The performance of these estimators can be evaluated using the bias and the *mean square error* (MSE) of the estimated coefficients which are given by

$$Bias(\hat{\xi}_m) = \frac{1}{M}\sum_{m=1}^{M}\left|\hat{\xi}_m - \xi\right|, \tag{26}$$

and

$$MSE(\hat{\xi}_m) = \frac{1}{M}\sum_{m=1}^{M}\left(\hat{\xi}_m - \xi\right)^2, \tag{27}$$

where $\hat{\xi}_m$ is the estimated value of the considered parameter ξ at the mth repetition out of M replicates. The computation of the simulation study is developed using R programming.

The simulation setting is as follows:

- Generate the response variable y_t from beta distribution $y_t \sim Beta(\mu_t, \varphi)$, with $\varphi = 1.5, 5$ and $\mu_t = \frac{e^{X_t'\beta}}{1+e^{X_t'\beta}}$ for $t = 1,2, \dots, n$.
- Generate two random variables x_1, and x_2 from standard uniform distribution U (0,1) for different sample sizes n=25, 50, and 100.
- Generate randomly different percentage of outliers such that $\omega\%$ =0%, 10%, 30% in y-direction to investigate the robustness of the MLE, S, MM, LTA, and REWLSE in the presence of outliers.
- The regression parameters are assumed to be $\beta_0 = \beta_1 = \beta_2 = 1$.
- All simulation results are repeated 1000 times and all the results of all separate experiments are obtained by the same series of random numbers.

The results of the estimated mean, bias, and the MSE of the parameters of beta regression model when the case of outliers ω% = 0%, 10%, 30% and different sample sizes using the MLE, S-estimation, MM-estimation, LTA estimation, and REWLSE methods are reported in Tables 1-6. Also, Figures 1-4 display the estimated MSE for different estimators with different percentages of outliers.

Table 1: Estimated Mean, Bias, and MSE values when $n = 25$ and $\varphi = 1.5$

Outliers	Method		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\varphi}$
$\omega = 0\%$	MLE	Mean	1.4370	1.0836	1.0847	1.8951
		Bias	0.4370	0.0836	0.0847	0.3951
		MSE	1.5192	1.2738	1.2665	1.4154
	S	Mean	1.2605	1.0167	1.0170	1.6026
		Bias	0.2605	0.0167	0.0170	0.1026
		MSE	1.3283	1.1114	1.1179	1.1951
	MM	Mean	1.2492	1.0134	1.0139	1.5762
		Bias	0.2492	0.0134	0.0139	0.0762
		MSE	1.2997	1.0800	1.0804	1.1724
	LTA	Mean	1.2962	1.0343	1.0347	1.6315
		Bias	0.2962	0.0343	0.0347	0.1315
		MSE	1.3513	1.1397	1.1340	1.2284
	REWLSE	Mean	1.2208	1.0102	1.0115	1.5535
		Bias	0.2208	0.0102	0.0115	0.0535
		MSE	1.2742	1.0665	1.0693	1.1534
$\omega = 10\%$	MLE	Mean	1.5541	1.1837	1.1861	1.9848
		Bias	0.5541	0.1837	0.1861	0.4848
		MSE	1.6110	1.3447	1.3494	1.6252
	S	Mean	1.3401	0.0699	1.0756	1.7354
		Bias	0.3401	0.9301	0.0756	0.2354
		MSE	1.4070	1.1678	1.1669	1.3751
	MM	Mean	1.3167	1.0488	1.0485	1.7168
		Bias	0.3167	0.0488	0.0485	0.2168
		MSE	1.3870	1.1412	1.1415	1.3521
	LTA	Mean	1.3663	0.0858	1.0968	1.7528
		Bias	0.3663	0.9142	0.0968	0.2528
		MSE	1.4224	1.2091	1.2013	1.4025
	REWLSE	Mean	1.2986	1.0294	1.0298	1.7058
		Bias	0.2986	0.0294	0.0298	0.2058
		MSE	1.3664	1.1216	1.1226	1.3412
$\omega = 30\%$	MLE	Mean	1.7374	1.3104	1.3255	2.2314
		Bias	0.7374	0.3104	0.3255	0.7314
		MSE	1.7865	1.5391	1.5460	1.8961
	S	Mean	1.4770	1.1703	1.1761	1.8921
		Bias	0.4770	0.1703	0.1761	0.3921
		MSE	1.6074	1.3346	1.3305	1.6821
	MM	Mean	1.4582	1.1542	1.1561	1.8794
		Bias	0.4582	0.1542	0.1561	0.3794
		MSE	1.5750	1.3015	1.3005	1.6624
	LTA	Mean	1.4994	1.1925	1.1948	1.9251
		Bias	0.4994	0.1925	0.1948	0.4251
		MSE	1.6226	1.3634	1.3688	1.7051
	REWLSE	Mean	1.4301	1.1355	1.1370	1.8634
		Bias	0.4301	0.1355	0.1370	0.3634
		MSE	1.5574	1.2736	1.2736	1.6575

Table 2: Estimated Mean, bias, and MSE values when $n = 25$ and $\varphi = 5$

Outliers	Method		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\varphi}$
$\omega = 0\%$	MLE	Mean	1.3930	1.0589	1.0597	5.6230
		Bias	0.3930	0.0589	0.0597	0.6230
		MSE	1.4247	1.2362	1.2388	1.3721
	S	Mean	1.2292	1.0105	1.0109	5.1853
		Bias	0.2292	0.0105	0.0109	0.1853
		MSE	1.2694	1.0779	1.0776	1.1664
	MM	Mean	1.1945	1.0079	1.0072	5.1567
		Bias	0.1945	0.0079	0.0072	0.1567
		MSE	1.2366	1.0557	1.0562	1.1458
	LTA	Mean	1.2591	1.0170	1.0163	5.2101
		Bias	0.2591	0.0170	0.0163	0.2101
		MSE	1.2906	1.0999	1.1001	1.1842
	REWLSE	Mean	1.1762	1.0057	1.0059	5.1491
		Bias	0.1762	0.0057	0.0059	0.1491
		MSE	1.2104	1.0343	1.0349	1.1412
$\omega = 10\%$	MLE	Mean	1.5191	1.1490	1.1522	5.7461
		Bias	0.5191	0.1490	0.1522	0.7461
		MSE	1.5243	1.3010	1.3001	1.5861
	S	Mean	1.3164	1.0423	1.0409	5.2954
		Bias	0.3164	0.0423	0.0409	0.2954
		MSE	1.3540	1.1444	1.1459	1.3458
	MM	Mean	1.2808	1.0226	1.0247	5.2785
		Bias	0.2808	0.0226	0.0247	0.2785
		MSE	1.3350	1.1279	1.1296	1.3221
	LTA	Mean	1.3395	1.0653	1.0601	5.3200
		Bias	0.3395	0.0653	0.0601	0.3200
		MSE	1.3747	1.1642	1.1611	1.3625
	REWLSE	Mean	1.2641	1.0096	1.0102	5.2696
		Bias	0.2641	0.0096	0.0102	0.2696
		MSE	1.3146	1.1072	1.1013	1.3192
$\omega = 30\%$	MLE	Mean	1.7042	1.2777	1.2852	5.9123
		Bias	0.7042	0.2777	0.2852	0.9123
		MSE	1.7345	1.4791	1.4864	1.8523
	S	Mean	1.4211	1.1401	1.1459	5.4853
		Bias	0.4211	0.1401	0.1459	0.4853
		MSE	1.5603	1.2914	1.2873	1.6422
	MM	Mean	1.3982	1.1240	1.1249	5.4631
		Bias	0.3982	0.1240	0.1249	0.4631
		MSE	1.5487	1.2587	1.2577	1.6283
	LTA	Mean	1.4591	1.1645	1.1625	5.5023
		Bias	0.4591	0.1645	0.1625	0.5023
		MSE	1.5923	1.3193	1.3249	1.6631
	REWLSE	Mean	1.3777	1.1098	1.1063	5.4581
		Bias	0.3777	0.1098	0.1063	0.4581
		MSE	1.5222	1.2388	1.2385	1.6192

Table 3: Estimated Mean, bias, and MSE values when $n = 50$ and $\varphi = 1.5$

Outliers	Method		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\varphi}$
$\omega = 0\%$	MLE	Mean	1.4186	1.0623	1.0628	1.7725
		Bias	0.4186	0.0623	0.0628	0.2725
		MSE	1.4769	1.2450	1.2426	1.3961
	S	Mean	1.2487	1.0137	1.0145	1.5901
		Bias	0.2487	0.0137	0.0145	0.0901
		MSE	1.2977	1.0864	1.0873	1.1725
	MM	Mean	1.2270	1.0119	1.0127	1.5723
		Bias	0.2270	0.0119	0.0127	0.0723
		MSE	1.2675	1.0687	1.0681	1.1563
	LTA	Mean	1.2721	1.0162	1.0190	1.6112
		Bias	0.2721	0.0162	0.0190	0.1112
		MSE	1.3210	1.1094	1.1065	1.2058
	REWLSE	Mean	1.2092	1.0098	1.0096	1.5686
		Bias	0.2092	0.0098	0.0096	0.0686
		MSE	1.2437	1.0474	1.0471	1.1402
$\omega = 10\%$	MLE	Mean	1.5244	1.1598	1.1580	1.8621
		Bias	0.5244	0.1598	0.1580	0.3621
		MSE	1.5762	1.3234	1.3256	1.6036
	S	Mean	1.3094	1.0348	1.0324	1.7152
		Bias	0.3094	0.0348	0.0324	0.2152
		MSE	1.3785	1.1572	1.1596	1.3665
	MM	Mean	1.2832	1.0106	1.0121	1.6931
		Bias	0.2832	0.0106	0.0121	0.1931
		MSE	1.3560	1.1230	1.1206	1.3454
	LTA	Mean	1.3329	1.0626	1.0603	1.7336
		Bias	0.3329	0.0626	0.0603	0.2336
		MSE	1.3954	1.1833	1.1819	1.3852
	REWLSE	Mean	1.2661	1.0088	1.0079	1.6873
		Bias	0.2661	0.0088	0.0079	0.1873
		MSE	1.3329	1.1009	1.0975	1.3381
$\omega = 30\%$	MLE	Mean	1.7123	1.2894	1.2923	2.1054
		Bias	0.7123	0.2894	0.2923	0.6054
		MSE	1.7663	1.5054	1.5138	1.8756
	S	Mean	1.4436	1.1403	1.1432	1.8861
		Bias	0.4436	0.1403	0.1432	0.3861
		MSE	1.5786	1.3184	1.3191	1.6632
	MM	Mean	1.4248	1.1213	1.1290	1.8657
		Bias	0.4248	0.1213	0.1290	0.3657
		MSE	1.5533	1.2889	1.2919	1.6441
	LTA	Mean	1.4720	1.1780	1.1712	1.9058
		Bias	0.4720	0.1780	0.1712	0.4058
		MSE	1.6070	1.3427	1.3587	1.6822
	REWLSE	Mean	1.4061	1.0957	1.1028	1.8592
		Bias	0.4061	0.0957	0.1028	0.3592
		MSE	1.5389	1.2655	1.2692	1.6382

Table 4: Estimated Mean, bias, and MSE values when $n = 50$ and $\varphi = 5$

Outliers	Method		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\varphi}$
$\omega = 0\%$	MLE	Mean	1.3582	1.0293	1.0259	5.6056
		Bias	0.3582	0.0293	0.0259	0.6056
		MSE	1.3926	1.1933	1.1951	1.3569
	S	Mean	1.1713	1.0256	1.0068	5.1634
		Bias	0.1713	0.0256	0.0068	0.1634
		MSE	1.2437	1.0556	1.0561	1.1489
	MM	Mean	1.1423	1.0043	1.0049	5.1452
		Bias	0.1423	0.0043	0.0049	0.1452
		MSE	1.2130	1.0337	1.0340	1.1234
	LTA	Mean	1.2186	1.0095	1.0098	5.1961
		Bias	0.2186	0.0095	0.0098	0.1961
		MSE	1.2698	1.0776	1.0780	1.1694
	REWLSE	Mean	1.1275	1.0027	1.0028	5.1396
		Bias	0.1275	0.0027	0.0028	0.1396
		MSE	1.1984	1.0121	1.0128	1.1192
$\omega = 10\%$	MLE	Mean	1.4927	1.1230	1.1257	5.7223
		Bias	0.4927	0.1230	0.1257	0.7223
		MSE	1.4993	1.2877	1.2829	1.5632
	S	Mean	1.2032	1.0269	1.0246	5.2856
		Bias	0.2032	0.0269	0.0246	0.2856
		MSE	1.3367	1.1202	1.1279	1.3231
	MM	Mean	1.1838	1.0164	1.0188	5.2632
		Bias	0.1838	0.0164	0.0188	0.2632
		MSE	1.3106	1.1029	1.1013	1.3025
	LTA	Mean	1.2218	1.0423	1.0399	5.3033
		Bias	0.2218	0.0423	0.0399	0.3033
		MSE	1.3547	1.1440	1.1410	1.3451
	REWLSE	Mean	1.1696	1.0079	1.0071	5.2586
		Bias	0.1696	0.0079	0.0071	0.2586
		MSE	1.2929	1.0837	1.0890	1.2981
$\omega = 30\%$	MLE	Mean	1.6724	1.2004	1.2052	5.9023
		Bias	0.6724	0.2004	0.2052	0.9023
		MSE	1.6962	1.4477	1.4497	1.8312
	S	Mean	1.3883	1.1348	1.1312	5.4638
		Bias	0.3883	0.1348	0.1312	0.4638
		MSE	1.5098	1.2583	1.2597	1.6258
	MM	Mean	1.3583	1.1153	1.1104	5.4457
		Bias	0.3583	0.1153	0.1104	0.4457
		MSE	1.4774	1.2278	1.2295	1.6024
	LTA	Mean	1.4185	1.1515	1.1572	5.4852
		Bias	0.4185	0.1515	0.1572	0.4852
		MSE	1.5355	1.2864	1.2889	1.6458
	REWLSE	Mean	1.3164	1.1052	1.0994	5.4391
		Bias	0.3164	0.1052	0.0994	0.4391
		MSE	1.4594	1.2068	1.2039	1.5971

Table 5: Estimated Mean, bias, and MSE values when $n = 100$ and $\varphi = 1.5$

Outliers	Method		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\varphi}$
ω = 0%	MLE	Mean	1.3784	1.0395	1.0381	1.7524
		Bias	0.3784	0.0395	0.0381	0.2524
		MSE	1.4435	1.2134	1.2125	1.3765
	S	Mean	1.2258	1.0084	1.0081	1.5702
		Bias	0.2258	0.0084	0.0081	0.0702
		MSE	1.2796	1.0661	1.0667	1.1689
	MM	Mean	1.2031	1.0066	1.0064	1.5563
		Bias	0.2031	0.0066	0.0064	0.0563
		MSE	1.2477	1.0418	1.0421	1.1458
	LTA	Mean	1.2596	1.0119	1.0127	1.5962
		Bias	0.2596	0.0119	0.0127	0.0962
		MSE	1.2996	1.0809	1.0800	1.1864
	REWLSE	Mean	1.1870	1.0022	1.0021	1.5491
		Bias	0.1870	0.0022	0.0021	0.0491
		MSE	1.2293	1.0206	1.0200	1.1235
ω = 10%	MLE	Mean	1.4992	1.1323	1.1301	1.8452
		Bias	0.4992	0.1323	0.1301	0.3452
		MSE	1.5489	1.3063	1.3039	1.5845
	S	Mean	1.2502	1.0371	1.0307	1.6925
		Bias	0.2502	0.0371	0.0307	0.1925
		MSE	1.3498	1.1303	1.1389	1.3458
	MM	Mean	1.2289	1.0132	1.0139	1.6752
		Bias	0.2289	0.0132	0.0139	0.1752
		MSE	1.3279	1.1016	1.1022	1.3258
	LTA	Mean	1.2739	1.0528	1.0537	1.7125
		Bias	0.2739	0.0528	0.0537	0.2125
		MSE	1.3785	1.1662	1.1613	1.3695
	REWLSE	Mean	1.2183	1.0111	1.0112	1.6682
		Bias	0.2183	0.0111	0.0112	0.1682
		MSE	1.3045	1.0801	1.0799	1.3191
ω = 30%	MLE	Mean	1.6900	1.2796	1.2798	2.0783
		Bias	0.6900	0.2796	0.2798	0.5783
		MSE	1.7302	1.4783	1.4989	1.8569
	S	Mean	1.4030	1.1252	1.1233	1.8633
		Bias	0.4030	0.1252	0.1233	0.3633
		MSE	1.5553	1.2875	1.3006	1.6459
	MM	Mean	1.3868	1.1004	1.1059	1.8631
		Bias	0.3868	0.1004	0.1059	0.3631
		MSE	1.5278	1.2606	1.2708	1.6242
	LTA	Mean	1.4285	1.1555	1.1508	1.8896
		Bias	0.4285	0.1555	0.1508	0.3896
		MSE	1.5778	1.3282	1.3372	1.6635
	REWLSE	Mean	1.3776	1.0814	1.0865	1.8591
		Bias	0.3776	0.0814	0.0865	0.3591
		MSE	1.5043	1.2468	1.2488	1.6198

Table 6: Estimated Mean, bias, and MSE values when $n = 100$ and $\varphi = 5$

Outliers	Method		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\varphi}$
$\omega = 0\%$	MLE	Mean	1.3294	1.0213	1.0218	5.5732
		Bias	0.3294	0.0213	0.0218	0.5732
		MSE	1.3759	1.1907	1.1916	1.3364
	S	Mean	1.0684	1.0056	1.0051	5.1563
		Bias	0.0684	0.0056	0.0051	0.1563
		MSE	1.2223	1.0342	1.0346	1.1236
	MM	Mean	1.0482	1.0039	1.0037	5.1364
		Bias	0.0482	0.0039	0.0037	0.1364
		MSE	1.2038	1.0128	1.0129	1.1025
	LTA	Mean	1.0899	1.0073	1.0077	5.1723
		Bias	0.0899	0.0073	0.0077	0.1723
		MSE	1.2378	1.0572	1.0571	1.1456
	REWLSE	Mean	1.0442	1.0018	1.0013	5.1293
		Bias	0.0442	0.0018	0.0013	0.1293
		MSE	1.1828	1.0096	1.0097	1.0982
$\omega = 10\%$	MLE	Mean	1.4561	1.1091	1.1040	5.6951
		Bias	0.4561	0.1091	0.1040	0.6951
		MSE	1.4795	1.2673	1.2623	1.5482
	S	Mean	1.1826	1.0118	1.0145	5.2634
		Bias	0.1826	0.0118	0.0145	0.2634
		MSE	1.3151	1.1080	1.1119	1.3058
	MM	Mean	1.1632	1.0094	1.0092	5.2458
		Bias	0.1632	0.0094	0.0092	0.2458
		MSE	1.2919	1.0880	1.0856	1.2863
	LTA	Mean	1.2008	1.0336	1.0324	5.2869
		Bias	0.2008	0.0336	0.0324	0.2869
		MSE	1.3381	1.1255	1.1296	1.3268
	REWLSE	Mean	1.1487	1.0077	1.0074	5.2392
		Bias	0.1487	0.0077	0.0074	0.2392
		MSE	1.2707	1.0611	1.0660	1.2785
$\omega = 30\%$	MLE	Mean	1.6647	1.2593	1.2552	5.8952
		Bias	0.6647	0.2593	0.2552	0.8952
		MSE	1.6559	1.4262	1.4227	1.8184
	S	Mean	1.3577	1.0978	1.0995	5.4428
		Bias	0.3577	0.0978	0.0995	0.4428
		MSE	1.4795	1.2336	1.2356	1.6058
	MM	Mean	1.3330	1.0727	1.0750	5.4236
		Bias	0.3330	0.0727	0.0750	0.4236
		MSE	1.4596	1.2153	1.2154	1.5821
	LTA	Mean	1.3755	1.1224	1.1286	5.4622
		Bias	0.3755	0.1224	0.1286	0.4622
		MSE	1.5085	1.2719	1.2693	1.6234
	REWLSE	Mean	1.3265	1.0599	1.0577	5.4193
		Bias	0.3265	0.0599	0.0577	0.4193
		MSE	1.4324	1.1997	1.1992	1.5765

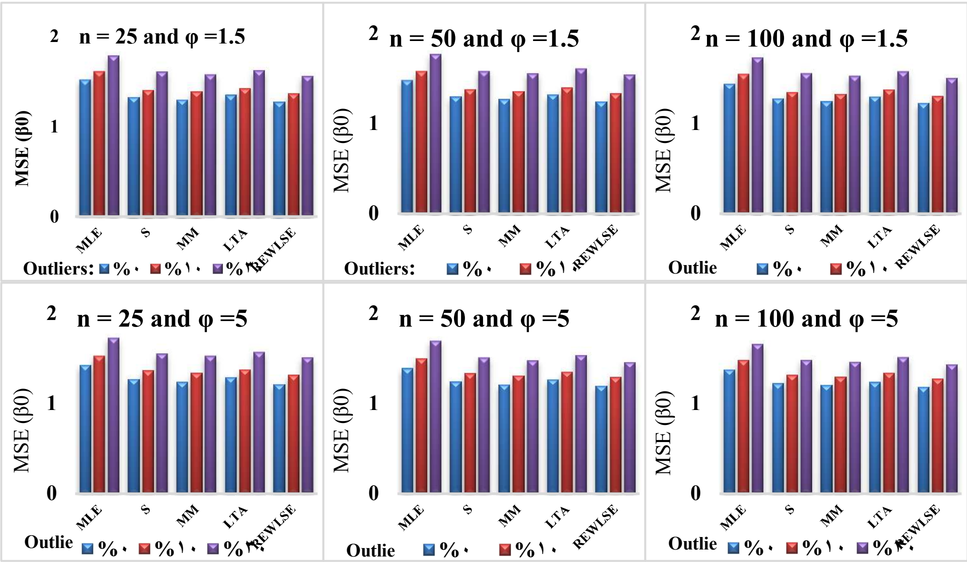

Figure 1: The MSE of different estimators for β_0 with different percentages of outliers

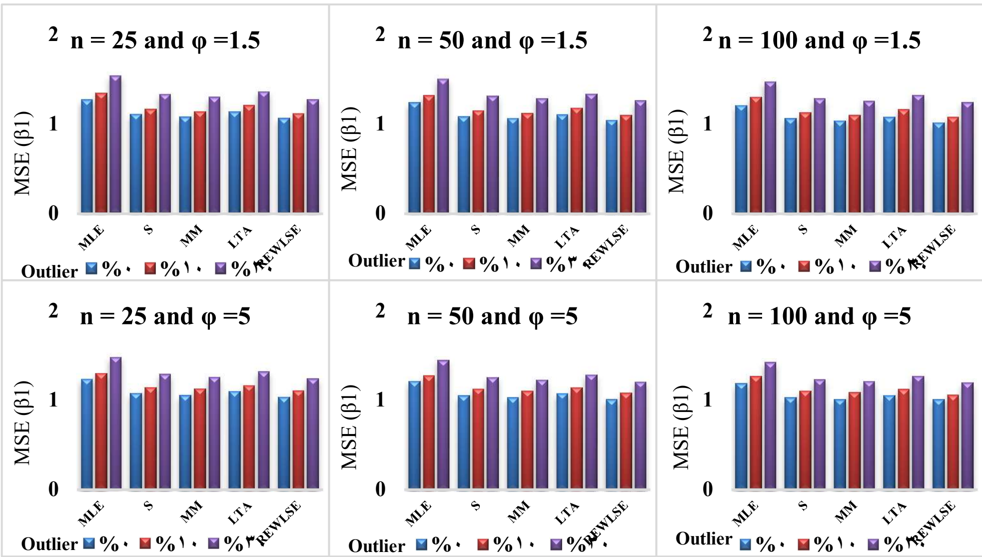

Figure 2: The MSE of different estimators for β_1 with different percentages of outliers

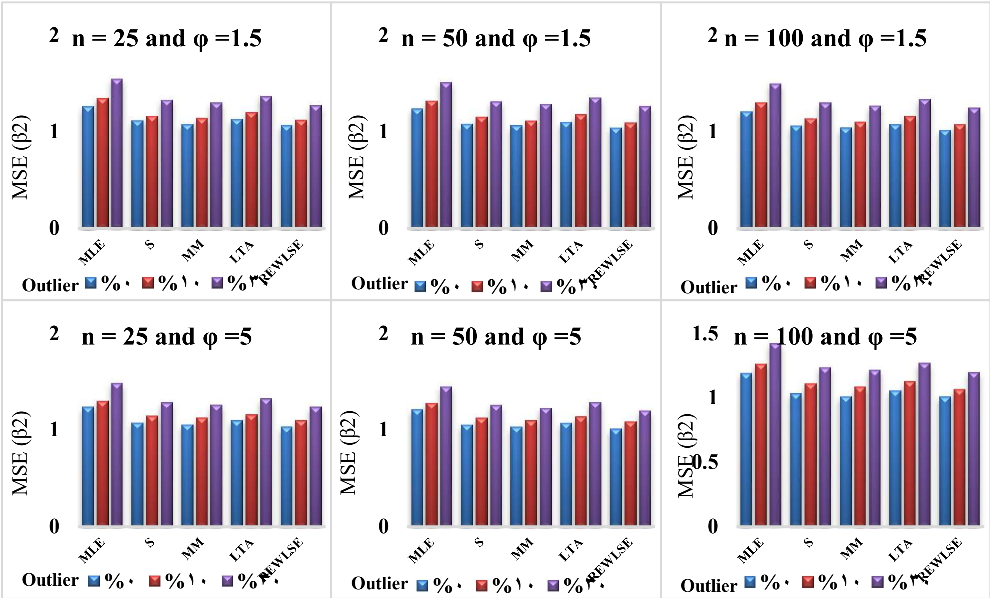

Figure 3: The MSE of different estimators for β_2 with different percentages of outliers

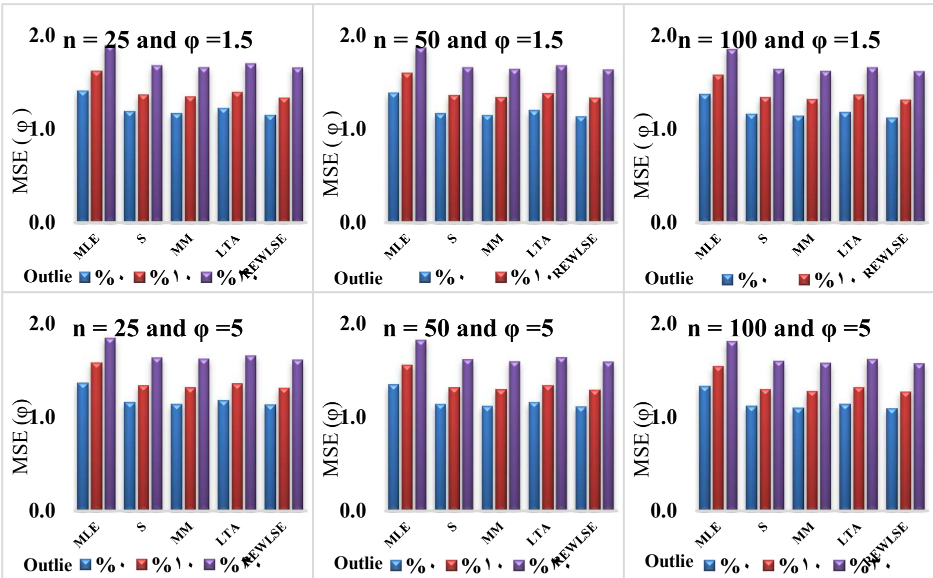

Figure 4: The MSE of different estimators for φ with different percentages of outliers

Tables 1-6 show that for all estimators, the estimated values of the bias and MSE decreases when the sample size n increases in all cases. However, when the dispersion parameter φ increases the bias and the MSE decreases for all estimators in all cases. Additionally, the estimated bias and MSE of all estimators are high with increasing the percentage values of outliers. It is also seen from Tables 1-6 and Figures 1-4 that the MLE has the largest estimated bias and MSE in all cases. Furthermore, the REWLSE gives the better performance compared to the S-estimator, MM-estimator, LTA estimator since it has the smallest values of bias and MSE for all levels of outliers corresponding to different values of n and φ.

6. Application

In this section, consider a subset of the data available in Monllor-Hurtado *et al.* (2017). The response variable (y) is the tropical tuna percentage and the explanatory variable (x) is the sea surface temperature. The data contains 77 observations in different points of the southern Indian Ocean in the year 2000. Figure 5 shows that the data contain outliers in the y-direction.

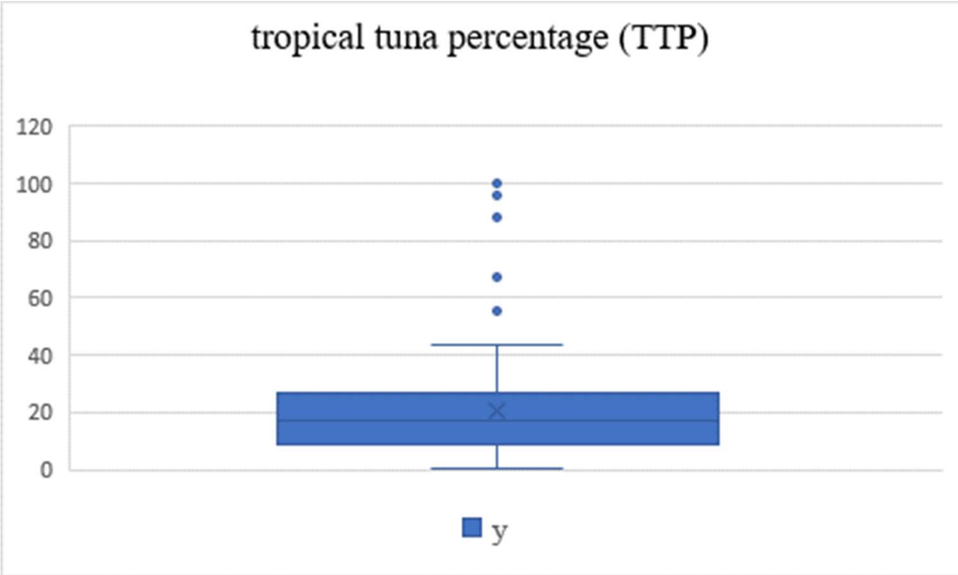

Figure 5: Outliers in the tropical tuna percentage (y)

The Kolmogorov test is taken into account for fitting the response variable to the beta distribution. The Kolmogorov test result with a test statistic equals 1.15 with p-value equals 0.071, validates the beta distribution applicability to this data. The estimated parameters and *standard error* (SE) of the estimators are showed in Table 7.

Table 7: The estimated parameters and SE of the estimators

Method	Parameters					
	β_0	SE	β_1	SE	φ	SE
MLE	-44.4282	10.1739	2.7989	0.4031	35.521	9.0654
S	-35.497	5.8172	2.1341	0.2131	25.324	4.158
MM	-32.465	4.7648	1.9744	0.1888	22.548	3.232
LTA	-39.550	6.5619	2.4786	0.2557	27.364	5.361
REWLSE	-30.851	4.564	1.8850	0.1685	22.214	3.145

From Table 7, it can be seen that when data contain outliers, the SE of the REWLSE is lower than the SE of the estimators: MLE, S-estimator, MM-estimator and LTA estimator. Therefore, the REWLSE has the better performance than all estimators.

7. Conclusions

In this paper, four robust estimation methods for beta regression model were suggested including the S-estimation, MM-estimation, LTA estimation, and REWLSE in the presence of outliers. A Monte Carlo simulation study and an application of tropical tuna percentage data set were provided to investigate the performance of the suggested methods. The results showed that the S-estimation, MM-estimation, LTA estimation, and the REWLSE methods gave better performance than the MLE in all considered cases. Additionally, the REWLSE ranked first with the lowest bias and MSE, then in the second rank was MM-estimator.

References

1. Abd El-Raoof, Z. M., El-Gohary, M. M., and Yehia, E. G. (2022). Robust Estimation for Weibull Regression Model. *The Scientific Journal of Commerce and Finance*, 42(3): 63-84.
2. Abonazel, M. R., Algamal, Z. Y., Awwad, F. A., and Taha, I. M. (2022). A New Two-Parameter Estimator for Beta Regression Model: Method, Simulation, and Application. *Frontiers Applied Mathematics and Statistics*, 7:1-10.
3. Alma, Ö. G. (2011). Comparison of Robust Regression Methods in Linear Regression. *International Journal of Contemporary Mathematical Sciences*, 6(9): 409-421.
4. Almetwally, E. M., and Almongy, H. M. (2018). Comparison between M-estimation, S-estimation, and MM-estimation Methods of Robust Estimation with Application and Simulation. *International Journal of Mathematical Archive*, 9(11): 55-63.
5. Bassett, G. W. (1991). Equivariant, Monotonic, 50% Breakdown Estimators, *The American Statistician*, 45: 135-137.
6. Beaton, A. E., and Tukey, J. W. (1974). The Firing of Power Series, Meaning Polynomials, Illustrated on Band-Spectroscopic data. *Technometrics,* 16: 147-185.
7. Çankaya, S., and Abacı, S. H. (2015). A Comparative Study of Some Estimation Methods in Simple Linear Regression Model for Different Sample Sizes in Presence of Outliers. *Turkish Journal of Agriculture-Food Science and Technology*, 3(6): 380-386.
8. Ferrari, S., and Cribari-Neto, F. (2004). Beta Regression for Modeling Rates and Proportions. *Journal of Applied Statistics*, 31: 799–815.
9. Gervini, D., and Yohai, V. J. (2002). A Class of Robust and Fully Efficient Regression Estimators. *The Annals of Statistics*, 30: 583-616.
10. Ghosh, A. (2019). Robust Inference Under the Beta Regression Model with Application to Health Care Studies. *Statistical Methods in Medical Research*, 28(3): 871-888.
11. Hawkins, M. D. and Olive, D. (1999). Applications and Algorithms for Least Trimmed Sum of Absolute Deviations Regression. *Computational Statistics and Data Analysis*, 32: 119-134.
12. Helmy, N., Shaaban, B., and Elgohary, M. (2021). Robust Mixture Regression Estimation Based on Least Trimmed Sum of Absolute Method by Using Several Models. *Al-Azhar Scientific Journal of the Commercial Faculties*, 26(1): 50-72.

13. Maluf, Y. S., Ferrari, S. L., and Queiroz, F. F. (2022). Robust Beta Regression through the Logit Transformation. *Statistic Methodology*, 15(1): 1-21.
14. Monllor-Hurtado, A., Pennino, M. G., and Sanchez-Lizaso, J. L. (2017). Shift in Tuna Catches Due to Ocean Warming. *Plos One*, 12(6): 1-10.
15. Ospina, R., Cribari-Neto, F., and Vasconcellos, K. L. (2006). Improved Point and Interval Estimation for A Beta Regression Model. *Computational Statistics and Data Analysis*, 51(2): 960-981.
16. Qasim, M., Månsson, K., and Kibria, B. M. G. (2021). On Some Beta Ridge Regression Estimators: Method, Simulation and Application. *Journal of Statistical Computation and Simulation*, 91(9): 1699-1712.
17. Rousseeuw, P. J., and Yohai, V. J. (1984). Robust Regression by Means of S-Estimators. *Robust and Nonlinear Time Series Analysis*, J. Frand, W. Hardle, and R. D. Martin (eds.), Lecture Notes in Statistics (26): 256-272, Springer-Verlag, New York.
18. Rousseeuw, P., and Leroy, A. (2003). *Robust Regression and Outlier Detection.* Wiley, New York.
19. Singgih, M. N. A., and Fauzan, A. (2022). Comparison of M-estimation, with MM-estimation to Get the Best Estimation of Robust Regression in Criminal Cases in Indonesia. *Jurnal Matematika, Statistika &Komputasi*, 18(2): 251-260.
20. Tableman, M. (1994a). The Influence Functions for the Least Trimmed Squares and the Least Trimmed Absolute Deviations Estimators, *Statistics and Probability Letters*, 9: 329-337.
21. Tableman, M. (1994b). The Asymptotic of the Least Trimmed Absolute Deviations (LTAD) Estimator, *Statistics and Probability Letters*, 19: 387-398.
22. Tinungki, G. M. (2018). Overcoming the Existence of Extreme Outlier Data by Using Robust MM-method Based on the Objective Function of Tukey bisquare. *Science Nature*, 1(1): 22-32.
23. Vasconcellos, K. L., and Cribari-Neto, F. (2005). Improved Maximum Likelihood Estimation in A New Class of Beta Regression Models. *Brazilian Journal of Probability and Statistics*, 19: 13-31.
24. Yohai, V. J. (1987). High Breakdown Point and High Efficiency Robust Estimates for Regression. *The Annals of Statistics*, 15(2): 624-656.
25. Yu, C., and Yao, W. (2017). Robust Linear Regression: A Review and Comparison. *Communications in Statistics-Simulation and Computation*, 46(8): 6261-6282.