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A New Extended Alpha-power Transformation of Burr-Generalized Gamma with an Application to Income

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Abstract

This paper presents a new generalization of the Alpha-Power Transformation (APT) to obtain a new class of probability distributions, which is particularly beneficial for analyzing of lifetime data. A special sub-model of the proposed family is introduced and discussed. The parameters of the model are estimated using maximum-likelihood estimation. In this regard, simulation studies are performed to investigate the properties of the estimators for the parameters. To include, the application of the model is demonstrated using the real data set.

Keywords: *alpha power transformation; Bur XII distribution; new extended alpha power transformation; Gini coefficient.*

1. Introduction

The advancement of conventional distributions has become a common exercise in statistical theory for the last two decades. Designing a new distribution from a classical one, through adding an additional parameter and using different methods, has recently become widespread. The main aims of these developments is to extant the classical distributions to more flexible ones suitable for analyzing complex data sets.

Mahdavi and Kundu (2017) proposed a new method for introducing statistical distributions via the cumulative distribution function (cdf) given by

$$F_{APT}(x) = \frac{\alpha^{F(x)} - 1}{\alpha - 1} \quad \text{for} \quad \alpha > 0, \alpha \neq 1, x \in \mathbb{R} \tag{1}$$

They named (1) the alpha power transformation (APT) and they proposed the alpha power exponential (APE) distribution by taking the exponential baseline distribution. They studied the main properties as well as the parameters estimation of the proposed distribution.

Nassar et al. (2017) introduced the alpha power Weibull distribution and showed that the new distribution gives better modelling than another generalizations of the Weibull distribution for analyzing two real data sets. Dey et al. (2017) used the generalized exponential baseline distribution to obtain the alpha power generalized exponential (APGE) distribution. Nadarajah and Okorie (2018) derived a closed form expressions for moment properties of APGE distribution. Nassar et al. (2018) proposed a new method for generating distributions based on the idea of alpha power transformation introduced by Mahdavi and Kundu (2017). The new method can be applied to any distribution by inverting its quantile function as a function of alpha power transformation. Hassan et al. (2018) motivated a lifetime distribution by APT called alpha power transformed extended exponential (APTEE) distribution. The APTEE model contains new recent models as; alpha power transformed exponential and alpha power transformed Lindley distributions. Dey, Ghosh, et al. (2019) and Dey, Nassar, et al. (2019)

studied the cases when G(x) correspond to inverse Lindley and Lindley distributions, respectively.
Mead et al. (2019) studied the general mathematical properties of the APT family and considered the alpha power exponentiated Weibull distribution. Nassar et al. (2019) proposed a new generalization of the APT class called Marshall Olkin alpha power family. Nassar et al. (2020) discussed the estimation of the parameters of the APE distribution using nine methods of estimation. Ogunde et al. (2020) proposed a three-parameter family of distribution, called the alpha power transformed extended Burr II distribution. ElSherpieny and Almetwally (2022) introduced the exponentiated generalized alpha power family of distributions to extend several other distributions. They used the new family and develop a new distribution, called the exponentiated generalized alpha power exponential distribution. Baharith (2022) introduced a flexible distribution called alpha power Kumaraswamy–Burr III distribution (APKIII) for fitting COVID-19 data in Saudi Arabia. The Kumaraswamy–Burr III distribution developed by combining two well-known families, namely, the Alpha power transformation and Kumaraswamy generated family.

Elbatal et al. (2018) proposed a new power transformation to extend the existing distributions. The distribution function of the Elbatal et al. (2018)'s new alpha power transformed family of distributions is given by

$$G_{NAPT}(x;\boldsymbol{\theta}) = \frac{F(x;\varepsilon)\alpha^{F(x;\varepsilon)}}{\alpha} \quad \text{for} \quad \alpha,\varepsilon > 0, \alpha \neq 1 \tag{2}$$

where $\boldsymbol{\theta} = (\alpha,\varepsilon)$. Using $F(x;\varepsilon)$ as the cdf of the Weibull model, Elbatal et al. (2018) proposed a three-parameter new alpha power transformed Weibull (NAPTW) distribution. Ahmad (2018) proposed a new family of APT to construct a new class of lifetime distributions, called the Zubair-G family which has the cdf as follows

$$G(x;\alpha,\varepsilon) = \frac{e^{\alpha F(x;\varepsilon)^2} - 1}{e^{\alpha} - 1}, \quad \text{for} \quad \alpha,\varepsilon > 0, x \in \mathbb{R}. \tag{3}$$

Ahmad et al. (2018) proposed another method, the new extended alpha power transformation (NEAPT) family of distributions with cdf given by

$$\boldsymbol{G_{NEAPT}(x;\alpha,\xi)=\frac{\alpha^{F(x;\xi)}-e^{\alpha F(x;\xi)}}{\alpha-e^{\alpha}};\quad \text{for}\quad \alpha,\xi>0,\alpha\neq e,x\in R.} \tag{4}$$

Mandouh et al. (2022) proposed an additional parameter to a family of distribution functions to bring more flexibility to the given family. They called this new family of distribution the $(ExAPT)$ family. The cdf of the proposed family is defined by the following expression

$$G_{ExAPT}(x;\mathbf{\theta})=\frac{\alpha^{kW(F(x;\xi))}-1}{\alpha-1},\quad \text{for}\quad \alpha,\xi>0,\alpha\neq 1,x\in\mathbb{R}. \tag{5}$$

where $\mathbf{\theta}=(\alpha,\xi)^T$ and $k=[W(1)]^{-1}$. The pdf of the new family of distributions is given by

$$\boldsymbol{g_{ExAPT}(x;\theta)=\frac{1}{\alpha-1}log(\alpha)kf(x;\xi)\left(\frac{\partial W(F(x;\xi))}{\partial x}\right)\alpha^{kW(F(x;\xi))},\quad \text{for}\quad \alpha>0,\alpha\neq 1.} \tag{6}$$

The aim of this paper is to present a new distribution using the $ExAPT$ method introduced by Mandouh et al. (2022) for the purpose of obtaining a new motivation of this family. The new distribution in the class of α-power transformation method is flexible and could be used to analyze a wide class of data. The paper is outlined as follows In Section 2; a special sub-model of the proposed distribution is introduced. In Section 3, we propose the new distribution called ExAPT-BG. Maximum likelihood estimates of the model parameters are obtained in Section 4. Simulation study is conducted in Section 6. In Section 7 a real data set is applied. Finally, the concluding remarks are provided in Section 8.

2. Sub-Model Description

Burr distribution were first discussed by Burr in 1942 as a two-parameter family. An additional scale parameter was introduced by Tadikamalla (1980). He established the relationship between Burr Type XII distribution and several other distributions. Several authors considered different aspects of Burr Type XII distributions such Al-Yousef (2002) discussed the problem of estimating the parameters of a doubly truncated Burr distribution when truncation points are unknown. Olapade (2008) obtained the cumulative distribution function and the rth moment of the generalized distribution and established the distribution of some order statistics of the distribution.

The cumulative distribution function (cdf), probability density function (pdf) and hazard rate function (hrf) of the Burr XII distribution are respectively given by

$$F_{BXII}(x) = 1 - \left(1 + x^{\lambda}\right)^{-m}, \quad for \quad x; \lambda; k > 0. \tag{7}$$

Both λ and m are shape parameters. The probability density function (pdf) corresponding to (7) is

$$f_{BXII}(x) = \lambda m x^{\lambda-1}\left(1 + x^{\lambda}\right)^{-m-1}, \quad x \geq 0; \lambda, m > 0. \tag{8}$$

Another family of distribution that have been used to model income data is the generalized gamma (GG) distribution introduced by Stacy (1962). It provides a flexible family with a variety of shapes and hazard functions for modeling duration time. It includes the exponential, Weibull, gamma and Rayleigh distributions, among other distribution as special sub-models. It is suitable for modeling data with different types of hazard rate function: increasing, decreasing, bathtub and unimodal. The GG distribution has been used in many applications such as engineering, hydrology and survival analysis. The probability density function of the generalized gamma distribution GG (a, b, c) is given by

$$f(x)_{GG} = \frac{a}{c\,\Gamma(b)}\left(\frac{x}{c}\right)^{(ab-1)} e^{-\left(\frac{x}{c}\right)^{a}}, x \geq 0, a, b, c > 0, \tag{9}$$

where $\Gamma(.)$ is the gamma function, b and a are shape parameters, and c is a scale parameter.

3. The New Extended Alpha power Transformation (ExAPT-BG) Distribution

Recently, many methods have been developed to bring flexibility to standard probability distributions, to increase their areas of applications, and also to obtain better fits. This work focuses on the use of the method developed by Mandouh et al. (2022), termed $(ExAPT)$, to obtain a new distribution named Extended α-Power-Burr XII-Generalized gamma (ExAPT-BG) distribution.

Let the GG (a,1,1) distribution for the $F(\cdot)$ function and represent the cumulative distribution function of Burr XII (λ,m) distribution for the $W(\cdot)$ function with $d1 = 0$ and $d2 = \infty$ in Eq. (5). In this case, the cdf of the Extended α-Power-Burr XII-Generalized gamma (ExAPT-BG) distribution is given by

$$\boldsymbol{G(x;\alpha,a,\lambda,m)=\frac{\alpha^{\frac{1-\left[1+\left(1-e^{-x^{a}}\right)^{\lambda}\right]^{-m}}{1-2^{-m}}}-1}{\alpha-1},\quad \text{for}\quad \alpha,a,\lambda,m>0,\alpha\neq 1,x\in\mathbb{R},}\qquad (10)$$

The probability density function (pdf) corresponding to (10) is given by

$$g(x;\alpha,a,\lambda,m)=\frac{log(\alpha)}{\alpha-1}\frac{am\lambda}{(1-2^{-m})}x^{a-1}\left[1-e^{-x^{a}}\right]^{\lambda-1}\left[1+\left(1-e^{-x^{a}}\right)^{\lambda}\right]^{-m-1}e^{-x^{a}}\alpha^{\left[\frac{1-\left(1+\left(1-e^{-x^{a}}\right)^{\lambda}\right)^{-m}}{1-2^{-m}}\right]},$$
$$\text{for}\quad \alpha,a,\lambda,m>0,\alpha\neq 1,x\in\mathbb{R}.\qquad (11)$$

The (sf), (hrf), (rhrf), and (chrf) of *(ExAPT-BG)* distribution, are respectively, given by

$$S(x;\alpha,a,\lambda,m)$$
$$=1-G(x;\alpha,a,\lambda,m)$$

$$\boldsymbol{=\frac{\alpha}{\alpha-1}\left[1-\alpha^{\left(1-\left(1+\left(1-e^{-x^{a}}\right)^{\lambda}\right)^{-m}\right)^{-1}}\right]}\qquad (12)$$

$$\boldsymbol{h(x;\alpha,a,\lambda,m)=\frac{g(x;\alpha,a,\lambda,m)}{1-G(x;\alpha,a,\lambda,m)}=\frac{log(\alpha)\,am\lambda\left[1-e^{-x^{a}}\right]^{\lambda-1}\left[1+\left(1-e^{-x^{a}}\right)^{\lambda}\right]^{-m-1}e^{-x^{a}}\,\alpha^{\left[\frac{1-\left(1+\left(1-e^{-x^{a}}\right)^{\lambda}\right)^{-m}}{1-2^{-m}}\right]}}{(1-2^{-m})\left(\alpha-\alpha^{\frac{1-\left(1+\left(1-e^{-x^{a}}\right)^{\lambda}\right)^{-m}}{1-2^{-m}}}\right)}}\qquad (13)$$

$$\boldsymbol{r(x;\alpha,a,\lambda,m)=\frac{g(x;\alpha,a,\lambda,m)}{G(x;\alpha,a,\lambda,m)}=\frac{log(\alpha)\,am\lambda\left[1-e^{-x^{a}}\right]^{\lambda-1}\left[1+\left(1-e^{-x^{a}}\right)^{\lambda}\right]^{-m-1}e^{-x^{a}}\,\alpha^{\left[\frac{1-\left(1+\left(1-e^{-x^{a}}\right)^{\lambda}\right)^{-m}}{1-2^{-m}}\right]}}{(1-2^{-m})\left(\alpha^{\frac{1-\left(1+\left(1-e^{-x^{a}}\right)^{\lambda}\right)^{-m}}{1-2^{-m}}}-1\right)}}\qquad (14)$$

$$H(x;\alpha,a,\lambda,m) = -log\big(1 - G(x;\alpha,a,\lambda,m)\big) = -\,log\left(\frac{\alpha - \alpha^{\frac{1-\left(1+\left(1-e^{-x^{a}}\right)^{\lambda}\right)^{-m}}{1-2^{-m}}}}{\alpha - 1}\right) \qquad (15)$$

The plots of the density and hrf's of the (ExAPT-BG) distribution are given in Figures 1 and 2, for different values of parameters.

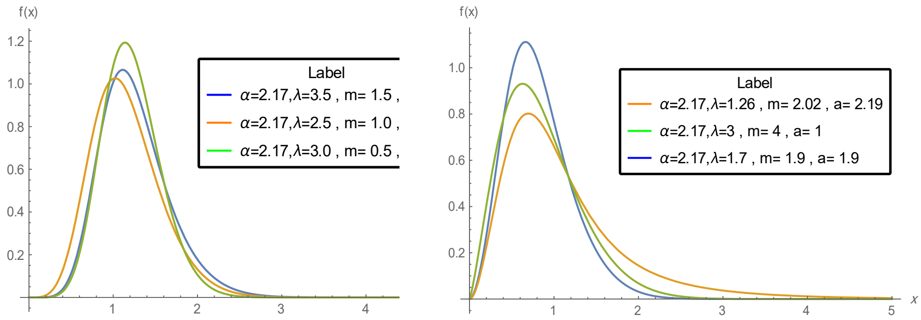

Figure 1: Plots of the ExAPT-BG pdf with different parameter values.

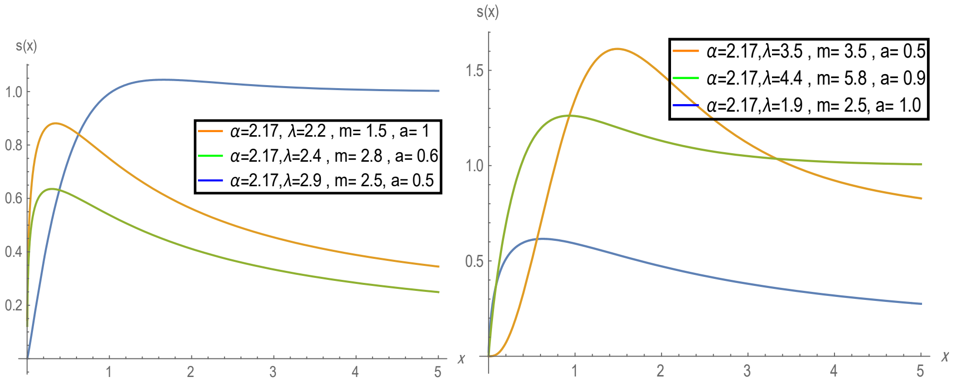

Figure 2: Plots of the ExAPT-BG hazard with different parameter values.

Alternative expression for the ExAPT-BG's pdf

This subsection provides expansion for the ExAPT-BG pdf given in Eq. (11). Employing the following series representation:

$$\alpha^{z}=\sum_{v=0}^{\infty}\frac{(log\alpha)^{v}}{v!}(z)^{v}. \tag{16}$$

Then, the ExAPT-BG's pdf can be expressed as

$$g(x;\alpha,a,\lambda,m)=\frac{am\lambda}{(\alpha-1)}x^{a-1}[1-e^{-x^{a}}]^{\lambda-1}\left[1+\left(1-e^{-x^{a}}\right)^{\lambda}\right]^{-m-1}e^{-x^{a}}\sum_{v=0}^{\infty}\frac{(log\alpha)^{v+1}}{v!}$$
$$\times\left(\frac{1}{1-2^{-m}}\right)^{v+1}\left[1-\left(1+\left(1-e^{-x^{a}}\right)^{\lambda}\right)^{-m}\right]^{v}, \tag{17}$$

Additionally, employing the following binomial theorem series for $q>0$ and $|z|<1$

$$(1-z)^{q}=\sum_{v_1=0}^{\infty}(-1)^{v_1}\binom{q}{v_1}z^{v_1} \tag{18}$$

Then, the ExAPT-BG's pdf will be reduced to

$$g(x;\alpha,a,\lambda,m)=\frac{am\lambda}{(\alpha-1)}x^{a-1}e^{-x^{a}}\left(1-e^{-x^{a}}\right)^{\lambda-1}\sum_{v=0}^{\infty}\sum_{v_1=0}^{\infty}(-1)^{v_1}\frac{(log\alpha)^{v+1}}{v!}$$
$$\times\left[\frac{1}{1-2^{-m}}\right]^{v+1}\binom{v}{v_1}\left[1+\left(1-e^{-x^{a}}\right)^{\lambda}\right]^{-mv_1-m-1}, \tag{19}$$

Using the binomial expansion once again, then the pdf in (19) can be written as

$$g(x;\alpha,a,\lambda,m)=\frac{am\lambda}{(\alpha-1)}x^{a-1}e^{-x^{a}}\sum_{v=0}^{\infty}\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}(-1)^{v_1+v_2}\binom{v}{v_1}\binom{mv_1+m+1}{v_2}$$
$$\times\frac{(log\alpha)^{v+1}}{v!}\left[\frac{1}{1-2^{-m}}\right]^{v+1}\left[1-e^{-x^{a}}\right]^{-\lambda(v_2+1)-1}, \tag{20}$$

Using the binomial expansion another time, then the pdf of ExAPT-BG's distribution is

$$g(x;\alpha,a,\lambda,m)=\frac{am\lambda}{(\alpha-1)}x^{a-1}e^{-x^{a}}\sum_{v=0}^{\infty}\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\sum_{v_3=0}^{\infty}(-1)^{v_1+v_2+v_3}\binom{v}{v_1}\binom{mv_1+m+1}{v_2}$$
$$\times\binom{\lambda(v_2+1)-1}{v_3}\frac{(log\alpha)^{v+1}}{v!}[\frac{1}{1-2^{-m}}]^{v+1}e^{-x^{a^{v_3}}}. \quad (21)$$

4. Quantile function

The quantile function (x_p) of the ExAPT-BG distribution can be expressed as

$$x_p=-log[1-([1-(1-2^{-m})\left(\frac{log(1+p(\alpha-1))}{log(\alpha)}\right)]^{\frac{-1}{m}}-1)^{\frac{1}{\lambda}}]^{\frac{1}{a}},\quad 0<p<1 \quad (22)$$

where $0<p<1$. From Equation (22), we can obtain the median (M) or the second quartile of ExAPT-BG distribution when $p=0.5$ as follows

$$M=-log[1-([1-(1-2^{-m})\left(\frac{log(0.5(1+\alpha))}{log(\alpha)}\right)]^{\frac{-1}{m}}-1)^{\frac{1}{\lambda}}]^{\frac{1}{a}},\quad \alpha>0,\alpha\neq 1. \quad (23)$$

We can obtain the first and third quartiles (p_1 and p_3) of ExAPT-BG distribution when $p=0.25$ and $p=0.75$ respectively, as follows

$$p_1=-log[1-([1-(1-2^{-m})\left(\frac{log(0.5(1.5+\alpha))}{log(\alpha)}\right)]^{\frac{-1}{m}}-1)^{\frac{1}{\lambda}}]^{\frac{1}{a}},\quad \alpha>0,\alpha\neq 1. \quad (24)$$

and

$$p_3=-log[1-([1-(1-2^{-m})\left(\frac{log(0.5(0.5+1.5\alpha))}{log(\alpha)}\right)]^{\frac{-1}{m}}-1)^{\frac{1}{\lambda}}]^{\frac{1}{a}},\quad \alpha>0,\alpha\neq 1. \quad (25)$$

From Equations (23, 24 and 25) first, second and third quartiles of ExAPT-BG distribution, we can obtain the Galton skewness (Sk), also known as Bowley's skewness that defined as

$$SK=\frac{p_3-2M+p_1}{p_3-p_1} \quad (26)$$

and also, kurtosis (Ku) measure which given as

$$Ku=\frac{x_{0.875}-x_{0.625}-x_{0.375}+x_{0.125}}{p_3-p_1} \quad (27)$$

5. Estimation

Let $X_1, X_2, \dots, X_n$ be a simple random sample of size n from the ExAPT-BG distribution, with α known, then from the pdf in Eq. (11) the likelihood function will be in the form

$$L(\lambda, a, m; \underline{x}) = \left(\frac{1}{\alpha-1}\right)^n (\log(\alpha))^n \left(\frac{1}{1-2^{-m}}\right)^n a^n \lambda^n m^n \prod_{i=1}^{n} \left(1-e^{-x_i^a}\right)^{\lambda-1} \prod_{i=1}^{n} x_i^{a-1}$$
$$\prod_{i=1}^{n} \left[1+\left(1-e^{-x_i^a}\right)^{\lambda}\right]^{-m-1} e^{-\sum_{i=1}^{n} x_i^a} \prod_{i=1}^{n} \left[\alpha^{\frac{1-\left[1+\left(1-e^{-x_i^a}\right)^{\lambda}\right]^{-m}}{1-2^{-m}}}\right]. \quad (28)$$

The log likelihood function is

$$l = n\log(\log(\alpha)) + n\log\left[\frac{1}{\alpha-1}\right] + n\log\left[\frac{1}{1-2^{-m}}\right] + n\log(a) + n\log(\lambda) + n\log(m) -$$
$$\sum_{i=1}^{n} x_i^a + (a-1)\sum_{i=1}^{n} \log(x_i) + (\lambda-1)\sum_{i=1}^{n} \log\left(1-e^{-x_i^a}\right) - (m+1)$$
$$\sum_{i=1}^{n} \log\left[1+\left(1-e^{-x_i^a}\right)^{\lambda}\right] + \log(\alpha)\sum_{i=1}^{n}\left[\frac{1-\left[1+\left(1-e^{-x_i^a}\right)^{\lambda}\right]^{-m}}{1-2^{-m}}\right]. \quad (29)$$

The maximum likelihood estimators of λ, a and m can be obtained by differentiating l with respect to λ, a and m and equating the results to zero.

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\hat{\lambda}} + \sum_{i=1}^{n} \log\left(1-e^{-x_i^{\hat{a}}}\right) - (\hat{m}+1)\sum_{i=1}^{n} \frac{\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}} \log\left(1-e^{-x_i^{\hat{a}}}\right)}{1+\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}}} +$$
$$\log(\alpha)\sum_{i=1}^{n} \frac{\hat{m}\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}} \log\left(1-e^{-x_i^{\hat{a}}}\right)\left[1+\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}}\right]^{-(1+\hat{m})}}{1-2^{-\hat{m}}} = 0, \quad (30)$$

$$\frac{\partial l}{\partial a} = \frac{n}{\hat{a}} - \sum_{i=1}^{n} \log(x_i) x_i^{\hat{a}} + \sum_{i=1}^{n} \log(x_i) + (\hat{\lambda}+1)\sum_{i=1}^{n} \frac{e^{-x_i^{\hat{a}}} x_i^{\hat{a}} \log(x_i)}{\left(1-e^{-x_i^{\hat{a}}}\right)}$$
$$-(\hat{m}+1)\sum_{i=1}^{n} \frac{\hat{\lambda}\, e^{-x_i^{\hat{a}}} \log x_i\, x_i^{\hat{a}} \left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}-1}}{1+\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}}} +$$
$$\log(\alpha)\sum_{i=1}^{n} \frac{\hat{m}\,\hat{\lambda}\, x_i^{\hat{a}} \log(x_i)\, e^{-x_i^{\hat{a}}} \left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}-1}\left[1+\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}}\right]^{-(1+\hat{m})}}{\left(1-2^{-\hat{m}}\right)} = 0, \quad (31)$$

and,

$$\frac{\partial l}{\partial m}=\frac{n}{\hat{m}}-\frac{n2^{-\hat{m}}log(2)}{1-2^{-\hat{m}}}-\sum_{i=1}^{n}log\left[1+\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}}\right]+$$
$$log(\alpha)\sum_{i=1}^{n}[\frac{\left[1+\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}}\right]^{-\widehat{ma}}log\left[1+\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}}\right]}{(1-2^{-\hat{m}})}-$$
$$\frac{2^{-\hat{m}}log(2)\left[1-\left(1+\left(1-e^{-x_i^{\hat{a}}}\right)^{\hat{\lambda}}\right)^{-\hat{m}}\right]}{(1-2^{-\hat{m}})^2}]=0. \qquad (32)$$

The system of the non-linear equations cannot be solved explicitly, so the maximum likelihood estimators of λ, a and m can be obtained by using a numerical technique.

6. Simulation study

A simulation study is carried out to asses the performance of the maximum likelihood estimation for (ExAPT-BG) by sample size n. The performance of the estimates for the parameters has been studied in term of their ; the biases and the mean square errors (MSEs) using the Mathematica 11 package. The numerical steps are listed as follows:

1. N=1000 random samples of sizes; $n = 30,50$ *and* 100 from the (ExAPT-BG) distribution are generated;
2. Selected initial guess values for the parameters are used;
3. The maximum likelihood estimates (mles) of (ExAPT-BG) model are evaluated for each sample size.
4. Calculate the biases and mean square error (MSE) of these estimators using Mathematica 11.

Some results of the simulation study are shown in the following Tables.

Table 1: The parameter estimation from (ExAPT-BG) distribution using maximum likelihood method.

n	par	Set 1 ($\alpha = 2.17, \lambda = 0.9; a = 1.6; m = 0.5$)			Set 2 ($\alpha = 2.17, \lambda = 0.9; a = 0.9; m = 0.5$)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	λ	0.8647	0.0012	0.1386	0.8524	0.0022	0.0599
	a	1.5902	0.0081	0.1164	0.8950	0.00002	0.0110
	m	0.4071	0.0086	3.8907	0.3209	0.0320	2.8895
50	λ	0.8768	0.00053	0.0890	0.8531	0.0021	0.0651
	a	1.5550	0.0030	0.0580	0.8958	0.00001	0.0081
	m	0.2277	0.0741	2.6605	0.4391	0.0037	3.4153
100	λ	0.9020	0.00004	0.0637	0.8974	0.0003	0.0287
	a	1.5267	0.0007	0.0326	0.9007	0.0004	0.0038
	m	0.4355	0.0041	1.5563	0.5056	0.00003	1.9364

Table 2: The parameter estimation from (ExAPT-BG) distribution using maximum likelihood method.

n	par	**Set 1** ($\boldsymbol{\alpha = 2.17, \lambda = 0.9; a = 0.9; m = 1.5}$)			**Set 2** ($\boldsymbol{\alpha = 2.17, \lambda = 0.9; a = 0.9; m = 2.5}$)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	λ	0.8523	0.0022	0.1147	0.8553	0.0012	0.0773
	a	0.8249	0.0056	0.0188	0.8354	0.0041	0.0148
	m	1.3419	0.0249	6.0847	2.2136	0.0819	4.213
50	λ	0.8684	0.0009	0.0980	0.8576	0.0010	0.0298
	a	0.8177	0.0067	0.0174	0.8385	0.0037	0.0119
	m	1.4416	0.0033	5.2820	2.4333	0.0044	3.2056
100	λ	0.8959	0.00001	0.0946	0.8963	0.0003	0.0249
	a	0.8190	0.0065	0.0126	0.8998	0.0003	0.0064
	m	1.4783	0.00046	4.4426	2.4852	0.0002	1.7408

Table 3: The parameter estimation from (ExAPT-BG) distribution using maximum likelihood method.

n	par	Set 1 ($\alpha = 2.17, \lambda = 0.7; a = 0.9; m = 1.2$)			Set 2 ($\alpha = 2.17, \lambda = 1.7; a = 1.2; m = 0.9$)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	λ	0.5601	0.0195	0.0238	1.6781	0.0004	0.2415
	a	0.5156	0.1477	0.1479	1.2105	0.0002	0.0276
	m	0.7348	0.2163	2.8611	0.8005	0.0098	2.9842
50	λ	0.6132	0.0075	0.0236	1.6889	0.0001	0.2349
	a	0.7636	0.0185	0.0199	1.2129	0.0002	0.0195
	m	0.8625	0.1139	1.3487	0.8566	0.0018	2.7693
100	λ	0.6515	0.0034	0.0136	1.7000	0.00001	0.1336
	a	0.9009	0.0020	0.0215	1.2117	0.0001	0.0102
	m	1.1351	0.0207	0.8674	0.8983	0.0002	1.5087

Table 4: The parameter estimation from (ExAPT-BG) distribution using maximum likelihood method.

n	par	Set 1 ($\alpha = 2.17, \lambda = 2.5; a = 0.5; m = 1.9$)			Set 2 ($\alpha = 2.17, \lambda = 3.0; a = 1.7; m = 1.3$)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	λ	2.3364	0.0267	0.9269	2.6333	0.1344	0.3367
	a	0.4616	0.0014	0.0158	1.6482	0.0219	0.0715
	m	1.5081	0.1535	4.9490	1.1332	0.5878	2.4094
50	λ	2.4558	0.0019	0.8903	2.7073	0.0856	0.2427
	a	0.4993	0.0003	0.0054	1.6575	0.0138	0.0455
	m	1.7416	0.0251	4.7414	1.1903	0.5036	1.4794
100	λ	2.5064	0.0001	0.7126	2.7360	0.0696	0.1353
	a	0.5001	0.0002	0.0044	1.6811	0.0065	0.0192
	m	1.8873	0.0001	4.0038	1.2744	0.3913	0.8347

The results show that, as it is usually the case, when the sample size n was increased, the mean square error of the estimates decrease which consistent with theoretical results. The MLE's of the parameters are approximately unbiased.

7. Application

A sample is taken from the Egyptian household income, expenditure and consumption survey (HIECS) for a period of time in $2015 \setminus 2016$. The original sample size is 11988 households and the sample taken was of 1200 households. The data represents the income of household in units of LE 10000. An attempt was made to fit the (ExAPT-BG) to the sample at hand. The computational software Mathematica 11 was used to solve the likelihood normal equations from Eq. (30) to Eq. (32). The descriptive statistics for the distribution of income are shown in the following table.

Descriptive statistics for the distribution of income in period $2015 \setminus 2016$

Obs.	**Min**	**Max**	**Mean**	**Var.**	**Skewness**	**Kurtosis**
1200	0.522	21.5435	4.0269	4.9840	2.2687	12.3323

Solving the likelihood normal equations yield the following results $\lambda = 12.4747$, $a = 0.7481$ and $m = 2.0275$. The cdf using these parameters and the empirical distribution of income household data can show how the data fits of the concerning distribution and Figure 3 shows how the two curves of the empirical and the cumulative distributions are almost identical.

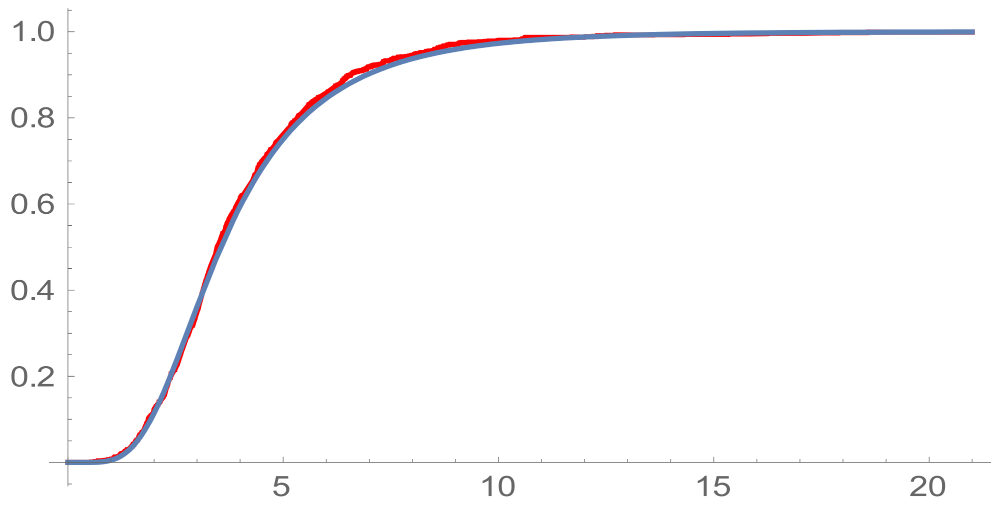

Figure 3: Empirical and Cumulative Distribution

The data is tested if it fits the distribution or not where the null hypothesis and its alternative will be:

H_0: The dataset follow the ExAPT-BG distribution.

against
H_1: The dataset do not follow the ExAPT-BG distribution.

After using different tests such as Kolmogorov Smirnov, Anderson Darling and Cramer Von Mises. The next table shows results of the different kinds of tests and test statistics for each test and also the p values:

The results of goodness of fit test, in the previous table, for the real data of income using different tests show that the null hypothesis the datasets have the same distribution cannot be rejected at the 5 percent of significance level.

Table 6: Results Using Different Goodness of Fit Tests

Test Name	Test statistics	pValue
Kolmogorov Smirnov	0.02	0.49
Anderson Darling	0.90	0.32
Cramer Von Mises	0.16	0.35

7.1 Income Inequality Measures

The Gini index is the Gini coefficient expressed as a percentage, and is equal to the Gini coefficient multiplied by 100. The Gini coefficient is equal to half of the relative mean difference (a measure of dispersion). The Gini coefficient is a measure of income inequality which is an indication of social welfare. It is defined as a ratio with values between 0 and 1. 0 corresponds to perfect income equality (i.e., everyone has the same income) and one corresponds to perfect income inequality (i.e. 1 person has all the income, while everyone else has zero income), see Dixon et al. (1988).

The Gini index can be expressed mathematically for a cumulative distribution function as shown in the following formula

$$G = 1 - \frac{1}{\mu}\int_0^{\infty} \left(1 - F(x)\right)^2 dx, \tag{33}$$

where μ is the mean of the distribution, see Dedduwakumara and Prendergast (2019).

Applying the previous Eq. (33) using the cumulative distribution of (ExAPT-BG) as in Eq. (10) and the parameters $\lambda = 12.4747$, $a = 0.7481$ and $m = 2.0275$ of the real income data of HIECS $2015 \setminus 2016$ survey, then the following equation

$$G = 1 - \frac{1}{\mu_1}\int_0^{\infty} \left(1 - \frac{e^{\frac{1-\left[1+\left(1-e^{-x^a}\right)^{\lambda}\right]^{-m}}{1-2^{-m}}} - 1}{e-1}\right)^2 dx, \tag{34}$$

where, $\mu_1 = \int_0^{\infty} \frac{am\lambda}{(e-1)(1-2^{-m})} x^a \left[1 - e^{-x^a}\right]^{\lambda-1} \left[1 + \left(1 - e^{-x^a}\right)^{\lambda}\right]^{-m-1} e^{\left[\frac{1-\left(1+\left(1-e^{-x^a}\right)^{\lambda}\right)^{-m}}{1-2^{-m}} - x^a\right]}$.

According to previous equation and these parameters, we get that Gini index equals 0.2810.

8. Concluding remarks

In this paper we have proposed a new four-parameter family of distribution, called the (ExAPT-BG) distribution. The proposed (ExAPT-BG) model has four shape parameters. The (ExAPT-BG) density function can take various forms depending on its shape parameters. The estimation of the model parameters through maximum likelihood method is discussed. A numerical evaluation is carried out to examine the performance of mles for (ExAPT-BG) model. Empirically, it is proved that the new model proposed in this study can provide a better fit to a real data set from the income of Egyptian households.

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