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Estimation of parameters of the exponentiated Pareto distribution using ranked set sampling

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Abstract

In this article, the maximum likelihood estimation method is used to estimate the unknown parameters of the exponentiated Pareto distribution based on ranked set sampling (RSS), median Ranked set sampling (MRSS) and simple random sample (SRS). Comparison between estimators of these techniques is made Through simulation study using some criteria: biases, mean square errors and the relative efficiency. At the same sample size, the study concluded that the estimators based on RSS and MRSS are more efficient than those obtained by the SRS.

Keywords: Maximum likelihood estimation; Expoentiated Pareto distribution; Ranked set sampling; Median ranked set sampling; the relative efficiency.

1. Introduction

The exponentiated Pareto (EP) distribution was introduced by Gupta et al. (1998). The probability density function (pdf) of EP distribution is given by

$$f(x) = \theta\lambda\left[1-(1+x)^{-\lambda}\right]^{\theta-1}(1-x)^{-(\lambda+1)}, x>0,\ \theta>0,\ \lambda>0 \tag{1}$$

Where θ and λ are shape parameters. The corresponding cumulative distribution function (cdf) is

$$F(x) = [1-(1+x)^{-\lambda}]^{\theta}, x>0,\ \theta>0,\ \lambda>0 \tag{2}$$

and the quantile function is given by

$$Q(u) = \left(\frac{1}{1-u^{1/\theta}}\right)^{\frac{1}{\lambda}} - 1 \qquad 0<u<1 \tag{3}$$

Ranked set sampling (RSS) is a technique of data collection based on the rank of the units of the drawn sample. This technique suggested by

McIntyre (1952) to estimate the mean of a pasture yield in Australia but he was not supported by Mathematical background. Takahasi and Wakimoto (1968) introduced the mathematical theory of the RSS and prove that the RSS mean estimator is more efficient than the mean of SRS because it has smaller variance. The efficiency of the RSS method may be decrease because of the error in the rank of the sample units, several modifications on the RSS were proposed by many authors. Extreme ranked set sampling (ERSS) introduced by Samawi et al. (1996), Muttlak, (1997,2003) suggested the median ranked set sampling (MRSS), The quartile ranked set sampling (QRSS) and the percentile ranked set sampling (PRSS) to estimate the population mean, Al-Saleh and Al-Kadiri (2000) presented the double ranked set sampling (DRSS) for estimating the population mean and they showed that the sample mean using DRSS is more efficient than the sample mean with SRS. Al-Saleh and Al-Omari (2002) proposed multistage ranked set sampling (MSRSS), Rashwan (2010) considered decile ranked set sampling (CRSS) to estimate the population mean and proved that the estimator using CRSS is more efficient than RSS and SRS for all symmetric and asymmetric distributions. Neoteric ranked set sampling (NRSS), as a new method for estimating the population mean and variance, proposed by Zamanzade and Al-Omari (2016).

In the literature, several authors interested in the parameters estimation of specific distributions using RSS and its modifications. Lam et al. (1994) obtained the estimation of unknown parameters of the two-parameter exponential distribution using RSS, Bhoj and Ahsanullah (1996) estimated the parameters of the generalized geometric distribution based on RSS, the method of RSS was used for estimating the parameters of bivariate exponential distribution by Chacko and Thomas (2007). Al-omari and Al-Hadhrami (2010) considered the estimation of parameters of modified weibull distribution, Abu-Dayyeh et al. (2013) used RSS for studying the estimation of the shape on the location parameters of the Pareto distribution and Hassan (2013) used maximum likelihood estimation and Bayes method for parameters estimation of the exponentiated exponential distribution. Khamnei and Mayan (2016) estimated the parameters of exponentiated Gumble distribution. Dey et al. (2017) studied estimation of Rayleigh distribution's parameters based on SRS, RSS and its modified versions, Esemen and Gurler (2018) considered the estimation of generalized Raylieh distribution using RSS and its modifications, Bantan et al. (2020) proposed a new distribution with three parameters called Zubair Lomax distribution and using maximum likelihood method to estimate the parameters of this distribution using RSS and, Sabry and Almetwally (2021) estimated exponential Pareto distribution parameters based on the maximum likelihood estimation method under RSS and DRSS.

In this paper, the Maximum likelihood estimation method is used to find an estimation of the unknown parameters of EP distribution using SRS, RSS and MRSS and the performance of the obtained estimators was examined using biases, mean square errors and the relative efficiency. In sections 2 and 3, RSS and MRSS are introduced and discussed. The maximum likelihood equations for estimating of the parameters of EP distribution based on SRS, RSS and MRSS are obtained and derived in section 4. In section 5, we present the results of the Monte Carlo simulation study to compare the efficiency of SRS, RSS and MRS. Finally; some conclusions are addressed in section6.

2 – Ranked Set Sampling

RSS is a useful statistical method for data collection and it improves the precision and increases the efficiency of estimation. In order to obtain a sample based on RSS, the following steps are required to be carried out

Step (1): Selecting randomly m samples (sets) of sizes m units from the population understudied.

Step (2): The m units in each sample are ranked, without measuring the variable, visually or by any inexpensive method with respect to the variable of interest.

Step (3): Choose from the first sample (set), the smallest ranked unit, from the second set, the second smallest Ranked unit and continuing in this manner until the largest ranked unit is selected from the m^{th} set.

Step (4): repeat the above steps from 1 to 3 c times (cycles) until obtaining a RSS sample of size n = mc units.

Let $X_{(i)j}$, i = 1, 2, …, m, j = 1, 2, …, c be a RSS drawn from EP distribution with sample size mc units and are independent. The pdf of $X_{(i)j}$ is the same the pdf of the i^{th} order statistic of a random sample $x_1, x_2, \ldots, x_n$, so the pdf of $X_{(i)j}$ is given by

$$g(X_{(i)j}) = \frac{m!}{(i-1)!\ (m-i)!} f(X_{(i)j})[F(X_{(i)j})]^{i-1}[1-F(X_{(i)j})]^{m-i} \tag{4}$$

3. Median ranked set sampling

MRSS was introduced by Muttlak (1997) to estimate the population mean and he showed that the MRSS provides an unbiased mean estimator and more efficient if compared to the SRS mean. The MRSS method may be summarized as follows:

Step (1): Randomly selected m samples each of size m units from the target population.

Step (2): The units of each sample (set) are ranked visually or any negligible cost method according to the variable of interest.

Step (3): From each sample in step (2), if the set size m is odd, choose the $(\frac{m+1}{2})^{th}$ ranked unit from all sets but if the set size m is even, select the $(\frac{m}{2})^{th}$ ranked unit from the first $\frac{m}{2}$ sets and select the $(\frac{m+2}{2})^{th}$ ranked unit from the remaining $\frac{m}{2}$ sets.

Step (4): repeat the above steps c cycles until obtaining MRSS sample of size n= mc units.

Let $X_{(i)j}$, i = 1, 2, …, m, j = 1, 2, …, c be a MRSS drawn from the EP distribution with sample size n = mc units. The pdf of $X_{(i)j}$ of the sample (set) size is even is given by

$$g(X_{(i)j}) = \frac{m!}{(\frac{m}{2})\left[\left(\frac{m}{2}-1\right)!\right]^2} f\left(X_{(i,\frac{m}{2})j}\right)\left[F(X_{(i,\frac{m}{2})j})\right]^{m/2-1}\left[1-F(X_{(i,\frac{m}{2})j})\right]^{m/2}$$

$$\times \frac{m!}{(\frac{m}{2})\left[(\frac{m}{2}-1)!\right]^2} f\left(X_{(i,\frac{m+2}{2})j}\right)\left[F(X_{(i,\frac{m+2}{2})j})\right]^{m/2}\left[1-F(X_{(i,\frac{m+2}{2})j})\right]^{\frac{m-2}{2}} \quad (5)$$

and if the set size is odd, the pdf of $X_{(i)j}$ is given by

$$g(X_{(i)j}) = \frac{m!}{\left[(\frac{m-1}{2})!\right]^2} \; f\left(X_{(i,\frac{m+1}{2})j}\right)\left[F(X_{(i,\frac{m+1}{2})j})\right]^{\frac{m-1}{2}}\left[1-F(X_{(i,\frac{m+1}{2})j})\right]^{\frac{m-1}{2}} \quad (6)$$

Where f(x) and F(x) are the pdf and cdf of the variable X.

4. Estimation using maximum likelihood method

In this section, the unknown parameters of the EP distribution are estimated using maximum likelihood method based on SRS, RSS and MRSS.

4.1. Estimation based on SRS

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n units from EP distribution with pdf in equation (1). The likelihood function (L_{SRS}) is

$$L_{SRS} = \prod_{i=1}^{n} \theta\lambda[1-(1+x_i)^{-\lambda}]^{\theta-1}[1+x_i]^{-(\lambda+1)} = (\theta\lambda)^n \prod_{i=1}^{n}[1-(1+x_i)^{-\lambda}]^{\theta-1}[1+x_i]^{-(\lambda+1)} \quad (7)$$

and the natural logarithm of likelihood function in equation (7) is given by

$$Ln\ L_{SRS} = n\ln\theta + n\ln\lambda + (\theta-1)\sum_{i=1}^{n} Ln[1-(1+x_i)^{-\lambda}] - (\lambda-1)\sum_{i=1}^{n} Ln(1+x_i) \quad (8)$$

Differentiating equation (8) with respect to θ and λ and equating zero, we obtain to the following equations

$$\frac{\partial LnL_{SRS}}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} Ln\left(1-(1+x_i)^{-\lambda}\right) \quad (9)$$

and

$$\frac{\partial LnL_{SRS}}{\partial \lambda} = \frac{n}{\lambda} + (\theta - 1)\sum_{i=1}^{n} \frac{(1+x_i)^{-\lambda} Ln(1+x_i)}{1-(1+x_i)^{-\lambda}} - \sum_{i=1}^{n} Ln(1+x_i) \tag{10}$$

Equations (9) and (10) are in explicit form, so they may be solved using numerical technique using iteration method to obtain ML estimators of θ and λ, say θ_{SRS} and λ_{SRS}.

4.2. Estimation based on RSS

Substituting in equation (4) by equations (1) and (2), the likelihood function is given by

$$L_{RSS} = \prod_{j=1}^{c}\prod_{i=1}^{m} \frac{m!}{(i-1)!(m-i)!}\, \theta\lambda\left[1-(1+x_{(i)j})^{-\lambda}\right]^{\theta i-1}[1+x_{(i)j}]^{-(\lambda+1)}\left[1-[1-(1+x_{(i)j})^{-\lambda}]^{\theta}\right]^{m-i}$$

$$= w^{mc}\ \theta^{mc}\ \lambda^{mc}\prod_{j=1}^{c}\prod_{i=1}^{m} \frac{m!}{(i-1)!(m-i)!}\, \theta\lambda\left[1-(1+x_{(i)j})^{-\lambda}\right]^{\theta i-1}[1+x_{(i)j}]^{-(\lambda+1)}\left[1-[1-(1+x_{(i)j})^{-\lambda}]^{\theta}\right]^{m-i} \tag{11}$$

Where $w = \frac{m!}{(i-1)!(m-i)!}$ .The natural logarithm of likelihood function ($Ln\, L_{RSS}$) is given by

$$LnL_{RSS} = mcLnw + mcLn\theta + mcLn\lambda + (\theta i - 1)\sum_{j=1}^{c}\sum_{i=1}^{m} Ln\left[1-(1+x_{(i)j})^{-\lambda}\right]$$
$$-(\lambda+1)\sum_{j=1}^{c}\sum_{i=1}^{m} ln(1+x_{(i)j}) + (m-i)\sum_{j=1}^{c}\sum_{i=1}^{m} ln\left(1-\left[1-(1+x_{(i)j})^{-\lambda}\right]^{\theta}\right) \tag{12}$$

The first derivates of LnL_{RSS} with respect to θ and λ, respectively, are given

$$\frac{\partial LnL_{RSS}}{\partial\theta}=\frac{mc}{\theta}+\sum_{j=1}^{c}\sum_{i=1}^{m} iLn\left[1-(1+x_{(i)j})^{-\lambda}\right]-\sum_{j=1}^{c}\sum_{i=1}^{m}(m-i)\frac{\left(1-(1+x_{(i)j})^{-\lambda}\right)^{\theta}Ln[1-(1+x_{(i)j})^{-\lambda}]}{1-\left(1-(1+x_{(i)j})^{-\lambda}\right)^{\theta}} \tag{13}$$

and

$$\frac{\partial LnL_{RSS}}{\partial\lambda}=\frac{mc}{\lambda}+\sum_{j=1}^{c}\sum_{i=1}^{m}(\theta i-1)\frac{(1+x_{(i)j})^{-\lambda}Ln(1+x_{(i)j})}{1-(1+x_{(i)j})^{-\lambda}}-\sum_{j=1}^{c}\sum_{i=1}^{m}Ln(1+x_{(i)j})$$

$$-\theta\sum_{j=1}^{c}\sum_{i=1}^{m}(m-i)\frac{\left[1-(1+x_{(i)j})^{-\lambda}\right]^{\theta-1}(1+x_{(i)j})^{-\lambda}Ln(1+x_{(i)j})}{1-\left(1-(1+x_{(i)j})^{-\lambda}\right)^{\theta}} \tag{14}$$

Setting $\frac{\partial \ln L_{RSS}}{\partial\theta}$ and $\frac{\partial \ln L_{RSS}}{\partial\lambda}$ equal zero and solving numerically these simultaneously to obtain estimates of θ and λ say θ_{RSS} and λ_{RSS}.

4.3. Estimation Based on MRSS

Using equation (5), the maximum likelihood function of the MRSS for EP data if the sample size is even is

$$L_{MRSSE}=\prod_{j=1}^{c}\prod_{i=1}^{m/2}\frac{m!}{\left[(\frac{m}{2}-1)!\right]^{2}\frac{m}{2}}\theta\lambda\left[1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}\right]^{\theta\frac{m}{2}-1}\left[1+x_{(i,\frac{m}{2})j}\right]^{-(\lambda+1)}\left[1-\left(1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}\right)^{\theta}\right]^{\frac{m}{2}}$$

$$\times\prod_{j=1}^{c}\prod_{i=\frac{m+2}{2}}^{m}\frac{m!}{\left[(\frac{m}{2}-1)!\right]^{2}\frac{m}{2}}\theta\lambda\left[1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}\right]^{\theta(\frac{m}{2}+1)-1}\left(1+x_{(i,\frac{m+2}{2})j}\right)^{-(\lambda+1)}\left[1-\left(1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}\right)^{\theta}\right]^{\frac{m}{2}-1}$$

$$=w_1^{mc}\ \theta^{mc}\ \lambda^{mc}\prod_{j=1}^{c}\prod_{i=1}^{m/2}\left[1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}\right]^{\theta\frac{m}{2}-1}[1+x_{(i,\frac{m}{2})j}]^{-(\lambda+1)}\left[1-\left(1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}\right)^{\theta}\right]^{\frac{m}{2}}$$

$$\times\prod_{j=1}^{c}\prod_{i=\frac{m+2}{2}}^{m}\left[1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}\right]^{\theta(\frac{m}{2}+1)-1}\left[1+x_{(i,\frac{m+2}{2})j}\right]^{-(\lambda+1)}\left[1-\left(1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}\right)^{\theta}\right]^{\frac{m}{2}-1} \tag{15}$$

Where $w_1 = \dfrac{m!}{\left[(\frac{m}{2}-1)!\right]^2 \frac{m}{2}}$. The log. Maximum likelihood function, say

LnL_{MRSSE} is given by

$$\begin{aligned}
LnL_{MRSSE} &= mcLnw_1 + mcLn\ \theta + mcLn\ \lambda + (\theta\frac{m}{2}-1)\sum_{j=1}^{c}\sum_{i=1}^{m/2} Ln[1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}] \\
&- (\lambda+1)\sum_{j=1}^{c}\sum_{i=1}^{m/2} Ln(1+x_{(i,\frac{m}{2})j}) + \frac{m}{2}\sum_{j=1}^{c}\sum_{i=1}^{m/2} Ln\left[1-\left(1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}\right)^{\theta}\right] \\
&+ [\theta(\frac{m}{2}+1)-1]\sum_{j=1}^{c}\sum_{i=\frac{m+2}{2}}^{m} Ln[1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}] - (\lambda+1)\sum_{j=1}^{c}\sum_{i=\frac{m+2}{2}}^{m} Ln[1+x_{(i,\frac{m+2}{2})j}] \\
&+ (\frac{m}{2}-1)\sum_{j=1}^{m}\sum_{i=\frac{m+2}{2}}^{m} Ln\left[1-\left(1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}\right)^{\theta}\right] \qquad (16)
\end{aligned}$$

Differentiating equation (16) with respect to θ and λ, respectively, we obtain

$$\begin{aligned}
\frac{\partial \ln L_{MRSSE}}{\partial \theta} &= \frac{mc}{\theta} + \frac{m}{2}\sum_{j=1}^{c}\sum_{i=1}^{m/2} Ln[1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}] - \frac{m}{2}\sum_{j=1}^{c}\sum_{i=1}^{m/2} \frac{\left(1-(1+x_{(i,m/2)j})^{-\lambda}\right)^{\theta} Ln[1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}]}{1-\left[1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}\right]^{\theta}} \\
&+ (\frac{m}{2}+1)\sum_{j=1}^{c}\sum_{i=\frac{m+2}{2}}^{m} Ln\left[1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}\right] - \frac{m}{2}\sum_{j=1}^{c}\sum_{i=\frac{m+2}{2}}^{m} \frac{[1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}]^{\theta} Ln(1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda})}{1-[1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}]^{\theta}}
\end{aligned}$$

(17)

and

$$\frac{\partial \ln L_{MRSSE}}{\partial\lambda}=\frac{mc}{\lambda}+(\theta\frac{m}{2}-1)\sum_{j=1}^{c}\sum_{i=1}^{m/2}\frac{(1+x_{(i,\frac{m}{2})j})^{-\lambda}Ln(1+x_{(i,m/2)j})}{1-(1+x_{(i,\frac{m}{2})j})^{-\lambda}}-\sum_{j=1}^{c}\sum_{i=1}^{m/2}Ln(1+x_{(i,\frac{m}{2})j})$$

$$-\theta\frac{m}{2}\sum_{j=1}^{m}\sum_{i=1}^{m/2}\frac{[1-(1+x_{(i,m/2)j})^{-\lambda}]^{\theta-1}(1+x_{(i,m/2)j})^{-\lambda}\ln(1+x_{(i,m/2)j})}{1-\left(1-(1+x_{(i,m/2)j})^{-\lambda}\right)^{\theta}}+$$

$$[\theta(\frac{n}{2}+1)-1]\sum_{j=1}^{c}\sum_{i=\frac{m+2}{2}}^{m}\frac{\left(1+x_{(i,\frac{m+2}{2}j}\right)^{-\lambda}Ln(1+x_{(i,\frac{m+2}{2})j})}{1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}}-\sum_{j=1}^{c}\sum_{i=\frac{m+2}{2}}^{m}(1+x_{(i,\frac{m+2}{2})j})$$

$$-\theta\left(\frac{m}{2}-1\right)\sum_{j=1}^{c}\sum_{i=\frac{m+2}{2}}^{m}\frac{\left[1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}\right]^{\theta-1}[1+x_{(i,\frac{m+2}{2})j}]^{-\lambda}\ln(1+x_{(i,\frac{m+2}{2})j})}{1-\left(1-(1+x_{(i,\frac{m+2}{2})j})^{-\lambda}\right)^{\theta}} \tag{18}$$

For odd set size m, the maximum likelihood function is given by using equation (6)

$$L_{MRSSO}=\prod_{j=1}^{c}\prod_{i=1}^{m}\frac{m!}{\left[(\frac{m-1}{2})!\right]^{2}}\theta\lambda\left[1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right]^{\theta(\frac{m+1}{2})-1}[1+x_{(i,\frac{m+1}{2})j}]^{-(\lambda+1)}\left[1-\left(1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right)^{\theta}\right]^{\frac{m-1}{2}}$$

$$=w_2^{mc}\ \theta^{mc}\ \lambda^{mc}\prod_{j=1}^{c}\prod_{i=1}^{m}\left[1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right]^{\theta(\frac{m+1}{2})-1}\left[1+x_{(i,\frac{m+1}{2})j}\right]^{-(\lambda+1)}\left[1-\left(1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right)^{\theta}\right]^{\frac{m-1}{2}} \tag{19}$$

Where $w_2=\dfrac{m!}{\left[(\frac{m}{2}-1)!\right]^2}$. The log. Likelihood function (Ln L_{MRSSO}) is given by

$$LnL_{MRSSO}=mcLnw_2+mcLn\ \theta+mcLn\ \lambda+\left[\theta(\frac{m+1}{2})-1\right]\sum_{j=1}^{c}\sum_{i=1}^{m}Ln\left[1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right]$$

$$-(\lambda+1)\sum_{j=1}^{c}\sum_{i=1}^{m}Ln(1+x_{(i,\frac{m+1}{2})j})+(\frac{m-1}{2})\sum_{j=1}^{c}\sum_{i=1}^{m}Ln\left[1-\left(1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right)^{\theta}\right] \quad (20)$$

Differentiate the equation (20) with respect to θ and λ, we get

$$\frac{\partial\ln L_{MRSSO}}{\partial\theta}=\frac{mc}{\theta}+(\frac{m+1}{2})\sum_{j=1}^{c}\sum_{i=1}^{m}Ln\left[1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right]-(\frac{m-1}{2})\sum_{j=1}^{c}\sum_{i=1}^{m}\frac{\left[1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right]^{\theta}Ln\left[1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right]}{1-\left(1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right)^{\theta}} \quad (21)$$

and

$$\frac{\partial\ln L_{MRSSO}}{\partial\lambda}=\frac{mc}{\lambda}+\left[\theta(\frac{m+1}{2})-1\right]\sum_{j=1}^{c}\sum_{i=1}^{m}\frac{(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}Ln(1+x_{(i,\frac{m+1}{2})j})}{1-\left(1+x_{(i,\frac{m+1}{2})j}\right)^{-\lambda}}-\sum_{j=1}^{c}\sum_{i=1}^{m}Ln(1+x_{(i,\frac{m+1}{2})j})$$

$$-\theta(\frac{m-1}{\lambda})\sum_{j=1}^{c}\sum_{i=1}^{m}\frac{\left((1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right)^{\theta-1}(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}Ln(1+x_{(i,\frac{m+1}{2})j})}{1-\left(1-(1+x_{(i,\frac{m+1}{2})j})^{-\lambda}\right)^{\theta}} \quad (22)$$

As there are no closed form for equations (19), (20), (21) and (22) under even and odd set size m respectively, the maximum likelihood estimators θ_{MRSS} and λ_{MRSS} are obtained numerically using iteration method.

5. Simulation Study

In this study, Monte Carlo simulation is done using R package software with N=1000 repetitions for different sample sizes, n=12,15,20,30 and different cycles number with two sets of initial values of parameters (θ=0.75 , λ=1.5 and θ=2 , λ=3). The comparisons between these estimators based on SRS, RSS and MRSS methods will be included bias, mean squared error (MSE) and the relative efficiency (RE) . These criteria were computed, respectively, as follows:

$$Bias(\hat{\beta}) = \frac{1}{1000}\sum_{h=1}^{1000}(\hat{\beta}_h - \beta) \quad ,$$

$$MSE(\hat{\beta}) = \frac{1}{1000}\sum_{h=1}^{1000}(\hat{\beta}_h - \beta)^2 \, ,$$

And
$$RE(\hat{\beta}_{SRS}, \hat{\beta}_{RSS\ methods}) = \frac{MSE(\hat{\beta}_{SRS})}{MSE(\hat{\beta}_{RSS\ methods})}$$

Where $\hat{\beta}_h$ is the estimated value of h^{th} repetition, h=1, 2, 3… 1000 and β is the population parameter. IF $RE(\hat{\beta}_{SRS}, \hat{\beta}_{RSS\ methods}) \geq 1$, then $\hat{\beta}_{RSS\ methods}$ is better than $\hat{\beta}_{SRS}$.

Simulation results were summarized in the following tables 1,2 and 3

Table1. Biases of the estimators of the EP distribution based on SRS, RSS and MRSS

(θ , λ)	n	SRS $\hat{\theta}$ $\hat{\lambda}$	m ,c	RSS $\hat{\theta}$ $\hat{\lambda}$	MRSS $\hat{\theta}$ $\hat{\lambda}$
(0.75,1.5)	12	0.5321 0.3521	3,4 4,3	0.3321 0.2504 0.3276 0.2336	0.2821 0.2223 0.2529 0.2167
	15	0.4936 0.2924	5,3 3,5	0.2727 0.2277 0.2401 0.2033	0.2529 0.2037 0.2433 0.1822
	20	0.3827 0.2313	10,2	0.2124 0.1935	0.1981 0.1601
	30	0.2732 0.2046	10,3 6,5	0.1767 0.1762 0.1798 0.1562	0.1629 0.1522 0.1511 0.1301
(2 , 3)	12	0.7021 0.4035	3,4 4,3	0.4542 0.3305 0.4276 0.2832	0.3229 0.2889 0.3101 0.2696
	15	0.6597 0.3819	5,3 3,5	0.4188 0.2666 0.3642 0.2216	0.2833 0.2413 0.2774 0.1935
	20	0.5298 0.3227	10,2	0.3304 0.2137	0.2557 0.1878
	30	0.4537 0.2824	10,3 6,5	0.2835 0.2025 0.2939 0.2101	0.1832 0.1555 0.1647 0.1466

Table2. MSEs of the estimators of the EP distribution based on SRS, RSS and MRSS

(θ , λ)	n	SRS $\hat{\theta}$ $\hat{\lambda}$	m ,c	RSS $\hat{\theta}$ $\hat{\lambda}$	MRSS $\hat{\theta}$ $\hat{\lambda}$
(0.75,1.5)	12	3.8315 2.7530	3,4 4,3	0.9310 0.6332 0.8203 0.5432	0.8521 0.5032 0.8225 0.4832
	15	2.3452 1.9321	5,3 3,5	0.8352 0.4361 0.8514 0.4129	0.7857 0.4323 0.7521 0.4001
	20	1.7290 1.1213	10,2	0.5432 0.3531	0.6245 0.3531
	30	1.1670 0.8533	10,3 6,5	0.3266 0.3131 0.4322 0.2925	0.5130 0.2634 0.4926 0.2732
(2 , 3)	12	4.2591 3.1222	3,4 4,3	2.8520 1.9352 2.2303 1.5370	2.5356 1.8315 2.6359 1.6216
	15	3.2591 2.7353	5,3 3,5	2.4367 1.6351 1.9398 1.5370	2.1567 1.4352 1.9354 1.3675
	20	2.7351 2.1351	10,2	1.8392 1.4356	1.5699 1.1356
	30	2.0335 1.6352	10,3 6,5	1.5376 1.2571 1.2356 0.9387	1.4320 0.9357 1.1564 0.8463

Table3. REs of the estimators of the EP distribution based on SRS, RSS and MRSS

(θ , λ)	n	m ,c	RSS		MRSS	
			$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
(0.75,1.5)	12	3,4	4.1154	4.2146	4.4965	5.4709
		4,3	4.6708	5.0681	4.6583	5.6974
	15	5,3	2.8079	4.4304	2.9846	4.4693
		3,5	2.7545	4.6793	3.1182	4.8290
	20	10,2	3.1829	3.1756	2.7686	3.1756
	30	10,3	2.2161	2.7253	2.2748	3.2395
		6,5	2.7001	2.9127	2.3691	3.1234
(2 , 3)	12	3,4	1.4779	1.6130	1.6797	1.7047
		4,3	1.9096	1.7115	1.6158	1.9253
	15	5,3	1.3375	1.6728	1.5112	1.7047
		3,5	1.6801	1.7796	1.6839	1.9253
	20	10,2	1.4871	1.4873	1.7422	1.8801
	30	10,3	1.3225	1.3007	1.4200	1.7475
		6,5	1.6457	1.7419	1.7584	1.9321

From tables 1,2 and 3 we noticed that,

- As the sample size increases, the bias and the MSEs decrease for all estimators based on SRS, RSS and MRSS.
- MSEs of all estimators based on RSS and MRSS are smaller than MSE of estimators based on SRS.
- MSE of all estimators based on MRSS is less than MSEs of estimators based on SRS and RSS.
- The RSS and MRSS estimates are more efficient than those estimates of SRS.
- The MRSS estimates are more efficient than those estimates of SRS and RSS.
- As initial values of θ and λ parameters are increase, the biases and MSEs of estimators based on SRS, RSS and MRSS are increase.

6. Conclusion

In this article, maximum likelihood estimation of unknown parameters of EP distribution based on SRS, RSS and MRSS methods are obtained and derived. The Monte Carlo simulation study is conducted to compare the performances of these methods on the parameters estimation of EP distribution. Comparison criteria included biases, MSE and RE. The results concluded that the RSS and MRSS estimates are more efficient than these estimates of the SRS method and the biases and the MSE of the estimates for parameters based on MRSS are smaller than the values of SRS and RSS. This means that, the estimation using MRSS is more efficient than estimation based on SRS and RSS.

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