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مارس ٢٠٢٢

Bayesian Estimation for Kumaraswamy Shanker Distribution with applications of COVID-19 data

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Abstact

In this paper, a new three-parameter distribution called the Kumaraswamy Shanker (Kw-Sh) distribution is proposed and studied. The reliability (survival) function, hazard rate function, reversed hazard rate function r and the cumulative hazard rate function of the new distribution is obtained. Some mathematical properties of this distribution such as moments, moments generating function, incomplete moments, quantile function, entropies (Renyi and Shannon) and mean deviation are derived. The method of maximum likelihood and Bayesian estimation method are used to estimate the distribution parameters. Finally, two real datasets, related to Covid-19, were used to examine the performance of the proposed distribution compared to Shanker and exponential distributions based on Akaike information criterion (AIC). The results showed that the new distribution is more proper in fitting data than other distributions. Also, the estimation of parameters using the Bayesian method is better than the maximum likelihood method

Keywords: Kumaraswamy distribution, Shanker distribution, moments generating function, quantile function, maximum likelihood estimation, Bayesian estimation.

Introduction:

The Shanker distribution is introduced by Shanker (2015) as a one – parameter lifetime distribution, $\theta > 0$. Its probability density function (pdf) is given by

$$g(x) = \frac{\theta^2}{\theta^2+1}(\theta + x)e^{-\theta x} \;\; ; x > 0,\; \theta > 0 \qquad (1)$$

The pdf in eq. (1) is a two-component mixture of an exponential distribution with scale parameter $\theta$and a gamma distribution with shape parameter 2 and the same scale parameter of exponential distribution

θas follows

$$g(x) = Pg_1(x) + (1 - P)g_2(x) \qquad (2)$$

Where $P = \frac{\theta^2}{\theta^2+1}$ is mixing proportion,$g_1(x) = \theta e^{-\theta x}$ is the pdf of exponential distribution and$g_2(x) = \theta^2 x e^{-\theta x}$ is the pdf of gamma distribution.

The cumulative distribution function (cdf) of Shanker distribution is given by

$$G(x) = 1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x} \; ; x > 0,\; \theta > 0 \qquad (3)$$

In the last few years, new generated families of continuous distributions have attracted many authors to develop new models. These families are obtained by entering one or more added shape parameter (s) to the baseline distribution.

The Kumaraswamy-G family (KW-G) is considered one of these families, which introduced by Kumaraswamy (1980) and Cordeiro and de Castro (2010). Nadarajah et al (2011)studied some statistical properties of this family. The Kumaraswamy distribution is not very common among statisticians and has been little exported in the literature. If G(x) is the baseline CDF of arandom variable X, Cordeiro and deCastro (2010) defined the CDF of the Kw-G distribution as

$$F(x) = 1 - \left[1 - \left(G(x)\right)^{a}\right]^{b} \quad (4)$$

Where $a > 0$ and $b > 0$ are shape parameters which govern skewness and tail weights. The pdf corresponding eq. (4) takes the following form

$$f(x) = ab\, g(x)(G(x))^{a-1}[1 - (G(x))^{a}]^{b-1} \quad (5)$$

Note that, the pdf in eq. (5) can be unimodal, increasing, decreasing or constant, depending on the parameter values. Nadarajah et al (2011),Several generalized distribution from eq.(5) have been defined and investigated in the literature including, for example, a new generalized Kumaraswamy distribution (Carrasco et al (2010)),the KW- Weibull distribution (Corderio et al. (2010)), the KW-pareto distribution (Bourguignon et al (2012)),the KW-double inverse exponential distribution (Aleem et al. (2013)),the KW-quasi Lindly distribution (Elbatal and Elgarhy (2013)),the Kw-generalized Rayleigh istribution (Gomes et al (2014)), the Kw-modified Weibull distribution (Cordeiro et al (2014)), a note on Kumaraswamy exponentiated distribution (Rashwan (2016)), The Kw-exponential Weibull distribution (Cordeiro et al (2016)),the Kw-Sushila distribution (Shawki and Elgarhy (2017)) and the Kw-exponentiated Fréchet distribution (Mansouir et al (2018)).

This paper offers new distribution called Kumaraswamy Shanker (KW-Sh) distribution. This study is organized as follows: In section 2, we propose and define the KW-Sh distribution and provide expansion for its density function. Some statistical properties of this distribution are discussed in section 3. In section 4, Maximum likelihood method and Bayesian method are used to estimate unknown parameters. Section 5 provides application to COVID-19 data sets. Finally, some conclusions are addressed in section 6.

2. Kumaraswamy Shanker distribution

In this section, we introduce the three-parameter Kumaraswamy Shanker (Kw-Sh) distribution. Using eq. (3) in eq. (4), The cdf of the Kw-Sh distribution is given by

$$F(x) = 1 - \left[1 - \left(1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1} e^{-\theta x}\right)^{a}\right]^{b} \quad , x > 0\,, \theta, a, b > 0 \qquad (6)$$

By differentiating eq. (6), we have

$$f(x)=\frac{ab\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}\left[1-\frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right]^{a-1}\left\{1-\left[1-\frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right]^{a}\right\}^{b-1};x,\theta,a,b>0 \qquad (7)$$

Where θ is scale parameter and a and b are shape parameters

The reliability (survival) function R (x), hazard rate function h(x), reversed hazard rate function r(x) and the cumulative hazard rate function H(x) of the Kw-Sh distribution are given by

$$R(x)=1-F(x)=\left[1-\left(1-\frac{(\theta^2+1)\quad+\theta x}{(\theta^2+1)}e^{-\theta x}\right)^{a}\right]^{b}$$

,

$$h(x)=\frac{f(x)}{R(x)}=\frac{ab\theta^2(\theta+x)e^{-\theta x}\left[1-\frac{(\theta^2+1)+\theta x}{(\theta^2+1)}e^{-\theta x}\right]^{a-1}}{(\theta^2+1)\left[1-\left(1-\frac{(\theta^2+1)+\theta x}{(\theta^2+1)}e^{-\theta x}\right)^{a}\right]}$$

,

$$r(x)=\frac{f(x)}{F(x)}=\frac{ab\theta^2(\theta+x)e^{-\theta x}\left[1-\frac{(\theta^2+1)+\theta x}{(\theta^2+1)}e^{-\theta x}\right]^{a-1}\left[1-\left(1-\frac{(\theta^2+1)+\theta x}{(\theta^2+1)}e^{-\theta x}\right)^{a}\right]^{b-1}}{(\theta^2+1)\left\{1-\left[1-\left(1-\frac{(\theta^2+1)+\theta x}{(\theta^2+1)}e^{-\theta x}\right)^{a}\right]^{b}\right\}}$$

, and

$$H(x)=-\ln R(x)=-ln\left[1-\left[1-\frac{(\theta^2+1)+\theta x}{(\theta^2+1)}e^{-\theta x}\right]^{a}\right]^{b}$$

Respectively. We notice that the following distributions are special cases of the Kw-Sh distribution

- If b=1, then eq. (7) gives exponentiated Shanker distribution with parameters a and θ.
- If a =b = 1, then eq. (7) give the Shanker distribution with θ parameter.

2.1 Expansion of theprobability density function

Here, we present a simple expansion for the pdf of Kw-Shanker distribution by using the generalized binomial theorem if β is positive and $|z|<1$, then

$$(1-z)^{\beta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}z^i\,,\quad \beta>0\,,|z|<1 \qquad (8)$$

The eq. (7) becomes

$$f(x)=\frac{ab\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left[1-\frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right]^{a(i+1)-1}$$

$$= \frac{ab\theta^2}{\theta^2+1}(\theta+x)\sum_{i,J=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\left(1+\frac{\theta x}{\theta^2+1}\right)^{j}e^{-\theta(j+1)x} \quad (9)$$

Then by using binomial theorem

$$(1+z)^j = \sum_{k=0}^{\infty}\binom{j}{k}z^k \qquad (10)$$

Using eq. (10) the eq. (9) becomes

$$f(x) = \frac{ab\theta^2}{\theta^2+1}(\theta+x)$$

$$\sum_{i,j,k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}$$

$$\binom{j}{k}\left(\frac{\theta x}{\theta^2+1}\right)^{k}e^{-\theta(j+1)x}$$

$$= ab\theta\sum_{i,j,k=0}^{\infty}\left(\frac{\theta}{\theta^2+1}\right)^{k+1}(-1)^{i+j}\binom{b-1}{i}$$
$$\binom{a(i+1)-1}{j}\binom{j}{k}(\theta x^k+x^{k+1})e^{-\theta(j+1)x}$$

Let $w_{ijk} =$
$ab\theta\left(\frac{\theta}{\theta^2+1}\right)^{k+1}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}$

Then the pdf of Kw-Shanker distribution can be rewritten as follows:

$$f(x) = \sum_{i,j,k=0}^{\infty} w_{ijk}\left(\theta x^k + x^{k+1}\right)e^{-\theta(j+1)x}, x>0, \;\; a,b,\theta>0 \qquad (11)$$

3. Statistical properties of Kw-Shanker distribution

In this section, we discuss some statistical properties of the Kw-Shanker distribution, specifically moments, moment generating function and quantile function.

3.1 Moments

Let X a random variable having the Kw-Sh distribution. using eq. (11), the r^{th} non-central moment of X can obtain as

$$E(X^r) = \sum_{i,j,k=0}^{\infty} w_{ijk} \int_0^{\infty} x^r (\theta x^k + x^{k+1}) e^{-\theta(j+1)x} dx$$

Let $y = \theta(j+1)x, y > 0,\ x = \frac{y}{\theta(j+1)}\ and\ dx = \frac{dy}{\theta(j+1)}$

$$\therefore\ E(X^r) = \sum_{i,j,k=0}^{\infty} w_{ijk} \int_0^{\infty} \left[\frac{\theta y^{r+k}}{\theta^{r+k+1}(j+1)^{r+k+1}} e^{-y} dy + \frac{y^{r+k+1}}{\theta^{r+k+2}(j+1)^{r+k+2}} e^{-y} dy\right]$$

$$E(X^r) = \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{r+k+1})}{(\theta(j+1))^{r+k+1}} + \frac{\Gamma(\mathrm{r+k+2})}{(\theta(j+1))^{r+k+2}}\right] \qquad (12)$$

Where $\Gamma(.)$ denotes the gamma function.

Substitution in the eq. (12) by r=1,2,3,4 we get the first four moments as follows

$$E(X) = \acute{\mu}_1 = \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{k+2})}{(\theta(j+1))^{k+2}} + \frac{\Gamma(\mathrm{k+3})}{(\theta(j+1))^{k+3}}\right]$$

$$E(X^2) = \acute{\mu}_2 = \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{k+3})}{(\theta(j+1))^{k+3}} + \frac{\Gamma(\mathrm{k+4})}{(\theta(j+1))^{k+4}}\right]$$

$$E(X^3) = \acute{\mu}_3 = \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{k+4})}{(\theta(j+1))^{k+4}} + \frac{\Gamma(\mathrm{k+5})}{(\theta(j+1))^{k+5}}\right]$$

$$E(X^4) = \acute{\mu}_4 = \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{k+5})}{(\theta(j+1))^{k+5}} + \frac{\Gamma(\mathrm{k+6})}{(\theta(j+1))^{k+6}}\right]$$

Based on the first four moments of the Kw-Sh distribution, the measures of mean (μ), variance (σ^2), skewens coefficient (SK) and kurtosis coefficient can be obtained as

$$\mu = E(x) = \acute{\mu}_1 = \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{k+2})}{\left(\theta(j+1)\right)^{k+2}} + \frac{\Gamma(\mathrm{k+3})}{\left(\theta(j+1)\right)^{k+3}}\right],$$

$$\sigma^2 = \acute{\mu}_2 - (\acute{\mu}_1)^2 \qquad , \qquad SK = \frac{\acute{\mu}_3 - 3\acute{\mu}_1\acute{\mu}_2 + 2(\acute{\mu}_1)^3}{\left(\acute{\mu}_2 - \acute{\mu}_1{}^2\right)^{3/2}}$$

and

$$Ku = \frac{\acute{\mu}_4 - 4\acute{\mu}_1\acute{\mu}_3 + 6\acute{\mu}_1{}^2\acute{\mu}_2 - 3\acute{\mu}_1{}^4}{\left(\acute{\mu}_2 - \acute{\mu}_1{}^2\right)^2}$$

3.2 Moment generating function

Now, we can derive the moments generating function $(mgf), M_x(t)$ for the Kw-Sh distribution as follows.

$$M_x(t) = E(e^{tx}) = \sum_{i,j,k=0}^{\infty} w_{ijk} \int_0^{\infty} e^{tx} (\theta x^k + x^{k+1}) e^{-\theta(j+1)x} dx$$

$$= \sum_{i,j,k=0}^{\infty} w_{ijk} \int_0^{\infty} (\theta x^k + x^{k+1})\, e^{x(\theta(j+1)-t)} dx$$

Based on the transformation $y = x(\theta(j+1) - t)$, we get

$$M_x(t) = \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{k}+1)}{(\theta(j+1)-t)^{k+1}} + \frac{\Gamma(\mathrm{k}+2)}{(\theta(j+1)-t)^{k+2}} \right] \tag{13}$$

In the same way, the factorial moment generating function, $M_x(\ln t)$, of the Kw-Sh distribution becomes

$$M_x(\ln t) = E\left(e^{x \ln t}\right)$$

$$= \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{k}+1)}{(\theta(j+1)-\ln t)^{k+1}} + \frac{\Gamma(\mathrm{k}+2)}{(\theta(j+1)-\ln t)^{k+2}} \right] \tag{14}$$

and the characteristic function of this distribution is given by

$$\phi_x(t) = E(e^{itx})) = \sum_{i,j,k=0}^{\infty} w_{ijk} \left[\frac{\theta\Gamma(\mathrm{k}+1)}{(\theta(j+1)-it)^{k+1}} + \frac{\Gamma(\mathrm{k}+2)}{(\theta(j+1)-it)^{k+2}} \right] \tag{15}$$

Where $i = \sqrt{-1}\ is\ the\ imaginary\ root$

3.3 Incomplete moments

The r^th incomplete moment, say $M_x(z)$ of the Kw. Sh distribution is given by

$$M_x(Z) = E(x^r \backslash X < z)$$

Using eq. (11), The $M_x(Z)$ can be obtained as follows

$$M_x(Z) = \sum_{i,j,k=\upsilon}^{\infty} W_{ijk} \int_0^{Z} x^r (\theta x^k + x^{k+1}) e^{-\theta(j+1)x} dx$$

Let $Y = \theta(j+1)x, 0 < Y < \theta\ (j+1)Z$,$x = \frac{Y}{\theta(j+1)}$ and $dx = \frac{dy}{\theta(j+1)}$

So, substituting in the above equation and integrating the above equation, we get

$$M_x(Z) = \sum_{i,j,k=0}^{\infty} W_{ijk} \left[\frac{\theta \lceil I(r+k+1, \theta(j+1)Z}{[\theta(j+1)]^{r+k+1}} + \frac{\lceil I(r+k+2, \theta(j+1)Z}{[\theta(j+1)]^{r+k+2}} \right] \quad (16)$$

Where $\lceil I(\emptyset)$ is incomplete gamma function and $\lceil I(\alpha, z) = \int_0^z y^{\alpha-1} e^{-y} dy$

The important application of the first incomplete moment is related to the Bonferroni and Lorenz curves. These curves are very useful in insurance, reliability, economics and medicine (Tahir et al., 2015).

3.4 Quantile function

The quantile function, $say\ Q(u) = F^{-1}(U) = x,$ of the kw-Sh distribution can be obtained by inverting eq. (6), we have the quantile function $Q(u)$as follows

$$\left(1 + \frac{\theta X_q}{\theta^2+1}\right) e^{-\theta X_q} = 1 - \left[1 - (1-U)^{\frac{1}{b}}\right]^{\frac{1}{a}} \quad (17)$$

We can easily generate X variable by Taking U as a uniform random variable in (0,1).

Using Q (u), we can derive the first quartile (Q_1), The median (Q_2) and the third quartile (Q_3) of the Kw-sh distribution by replacing u with the value 0.25. 0.50 and 0.75 in eq. (17).

3.5 Entropies

The Renyi entropy of a random is a measure of variation of the uncertainty. If X is a random variable with pdf, f(x), defined in eq, (7), then the Renyi entropy of x is given by (Song, 2001).

$$I_{X:R}(q) = \frac{1}{1-q} In(I_x(q))$$

Where $I_x(q) = \int_R (f(x))^q\, dx\ \ , q > 0\ and\ q \neq 1$
For Kw-Sh a random variable using eq. (7) , then

$$I_x(q) = \left(\frac{ab\theta^2}{\theta^2+1}\right)^q \int_0^{\infty} (\theta+x)^q e^{-\theta q x} \left[1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1} e^{-\theta x}\right]^{q(a-1)} \left[1 - \left[1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1} e^{-\theta x}\right]^a\right]^{q(b-1)}$$

Applying the binomial expansion and integrating the above equation, we get

$$I_x(\boldsymbol{q}) = (\boldsymbol{ab\theta})^{\boldsymbol{q}} \sum_{\boldsymbol{i,j,k=v}}^{\infty} \sum_{\boldsymbol{m=0}}^{\boldsymbol{q}} (-\boldsymbol{1})^{\boldsymbol{i+j}} \binom{\boldsymbol{q(b-1)}}{\boldsymbol{i}} \binom{\boldsymbol{a_i + q(a-1)}}{\boldsymbol{j}}$$

$$\binom{\boldsymbol{j}}{\boldsymbol{k}} \binom{\boldsymbol{q}}{\boldsymbol{m}} \left(\frac{\boldsymbol{\theta}}{\boldsymbol{\theta^2+1}}\right)^{\boldsymbol{k+q}} \boldsymbol{\theta^m} \int_{\boldsymbol{0}}^{\infty} \boldsymbol{x^{k+q-m} e^{-\theta(j+q)x} dx}$$

Let $= \theta(j+q)x$, y>0 ,$x = \frac{y}{\theta(j+q)}$ and $dx = \frac{dy}{\theta(j+q)}$

So, $\boldsymbol{I_x(q)} =$

$$(\mathrm{ab\theta})^{\mathrm{q}} \sum_{\mathrm{i,j,k=0}}^{\infty} \sum_{\mathrm{m=0}}^{\mathrm{q}} (-1)^{\mathrm{i+j}} \binom{\mathrm{q(b-1)}}{\mathrm{i}} \binom{\mathrm{a_i + q(a-1)}}{\mathrm{j}} \binom{\mathrm{j}}{\mathrm{k}} \binom{\boldsymbol{q}}{\boldsymbol{m}} \boldsymbol{\theta^m}$$

$$\left(\frac{\boldsymbol{\theta}}{\boldsymbol{\theta^2+1}}\right)^{\boldsymbol{k+q}} \frac{\boldsymbol{\Gamma(k+q-m+1)}}{[\boldsymbol{\theta(j+q)}]^{\boldsymbol{k+q-m+1}}}$$

Hence, the Renyi entropy becomes

$$I_{X:R}(q) = \frac{q \ ln \ ab\theta}{1-q} + (1-q)^{-1} Ln \sum_{\boldsymbol{i,j,k=0}}^{\infty} \sum_{\boldsymbol{m=0}}^{\boldsymbol{q}} (-\boldsymbol{1})^{\boldsymbol{i+j}} \binom{\boldsymbol{q(b-1)}}{\boldsymbol{i}} \binom{\boldsymbol{a_i + q(a-1)}}{\boldsymbol{j}}$$

$$\binom{\boldsymbol{j}}{\boldsymbol{k}} \binom{\boldsymbol{q}}{\boldsymbol{m}} \boldsymbol{\theta^m} \left(\frac{\boldsymbol{\theta}}{\boldsymbol{\theta^2+1}}\right)^{\boldsymbol{k+q}} \frac{\boldsymbol{\Gamma(k+q-m+1)}}{[\boldsymbol{\theta(j+q)}]^{\boldsymbol{k+q-m+1}}} \quad (\boldsymbol{18})$$

The Shannon entropy (Shannon1948), say E_{sh} of a random variable x with pdf in eq. (11) is defined by

$$E_{sh} = E[-lnf(x)]$$

$$= E\left[-In \sum_{i,j,k=0}^{\infty} W_{ijk} (\theta x^k + x^{k+1}) e^{-\theta(j+1)x}\right]$$

$$= -\left[In \sum_{i,j,k=0}^{\infty} W_{ijk} + In[\theta E(x^k) + E(x^{k+1})] - \theta(j+1)E(x)\right]$$

$$E_{sh} = \theta(j+1)E(x) - In \sum_{i,j,k=0}^{\infty} W_{ijk} - In[\theta E(x^k) + E(x^{k+1})] \qquad (19)$$

Where

$$E(x) = \sum_{i,j,k=0}^{\infty} W_{ijk} \left[\frac{\theta \Gamma(k+2)}{(\theta(j+1))^{k+2}} + \frac{\Gamma(k+3)}{(\theta(j+1))^{k+3}}\right]$$

and

$$E(x^k) = \sum_{i,j,k=0}^{\infty} W_{ijk} \left[\frac{\theta \Gamma(2k+1)}{\left(\theta(j+1)\right)^{2k+1}} + \frac{\Gamma(2k+2)}{\left(\theta(j+1)\right)^{2k+2}}\right]$$

Note that, entropies have been used in many applications in engineering and other sciences.

3.6 Mean deviation

The mean deviation is a measure of the dispersion derived by computing the mean of the absolute value of differences between the observed values of the variable and the mean or the median of this variable.

The mean deviation about the mean and the median are defined.

$$D(u) = E|X - \mu|$$
$$= \int_0^\infty |x - \mu_1|\, f(x)dx = 2\mu_1 F(\mu_1) - 2m_1(\mu_1) \qquad (20)$$

and

$$D(M) = \int_0^\infty |x - M|\, f(x)dx = \acute{\mu}_1 - 2m_1(M) \qquad (21)$$

respectively, where $\mu_1 = E(x)$ comes from equ. (12) where r=1, M is the median (x) comes from equ. (6) when $F(x) = \frac{1}{2}$, $F(\mu_1)$ comes from the cdf in equ. (6) and $m_1(Z) = \int_0^Z xf(x)dx$ is the first incomplete moment obtained from equ. (16) with r=1.

3.7 Order statistics

Order statistics plays an important role in many applied fields of statistics such as quality control and reliability. Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from the Kw-Sh distribution with parameters a, b and θ. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the corresponding order statistics. The density function of the i[th] order statistic $X_{1:n}$ say $f_{i:n}(x)$ for i=1,2,…,n is given by Arnold et al(2008)

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)}[F(x)]^{i-1}[1 - F(x)]^{n-i}$$

$$= \frac{f(x)}{B(i, n-i+1)} \sum_{L=0}^{n-i} (-1)^L \binom{n-i}{L} [F(x)]^{i+L-1}$$

Inserting equ.(6) and equ.(7) in the last equation, we obtain

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \frac{ab\theta^2}{\theta^2+1}(\theta + x)e^{-\theta x}\left[1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right]^{a-1}\left[1 - \left(1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right)^a\right]^{b-1}$$
$$\times \sum_{L=0}^{n-i} (-1)^L \binom{n-i}{L} \left[1 - \left[1 - \left(1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right)^a\right]^b\right]^{i+L-1}$$

$$= \frac{1}{B(i, n-i+1)} \sum_{L=0}^{n-i} \sum_{s=0}^{\infty} (-1)^{L+S} \binom{n-i}{L} \binom{i+L-1}{S}$$
$$\frac{ab(s+1)\theta^2}{(s+1)(\theta^2+1)}(\theta + x)e^{-\theta x} \times \left[1 - \frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right]^{a-1}$$

$$\left[1-\left(1-\frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x}\right)^a\right]^{b(S+1)-1}$$

$$f_{i:n}(x)=\frac{1}{B(i,n-i+1)}\sum_{L=0}^{n-i}\sum_{S=0}^{\infty}(-1)^{L+S}\binom{n-i}{L}\binom{i+L-1}{S}$$
$$f(x,a,b(S+1),\theta) \qquad (22)$$

Where the beta function is

$$B(i,n-i+1)=\frac{\lceil(i)\lceil)n-i+1)}{\lceil(n+1)}=(i-1)!\,(n-i)$$

and $f(x,a,b(S+1),\theta)$ is the pdf of Kw-Sh distribution with parameters $a, b(S+1)$ and θ respectively.

4. Estimation Methods

The method of maximum likelihood and Bayesian estimation method are used to estimate the distribution parameters

4.1 Maximum likelihood estimation

In this sub- section, the maximum likelihood estimates (MLE_s) of the parameters of the Kw-Sh distribution are derived.

Let $X_1, X_2, \dots, X_n$ be an independent random sample of size n from The Kw-Sh distribution. then the likelihood is defined as the joint density evaluated at x_1, x_2, … x_n .Parameters are selected so that the likelihood function is maximized. The log- likelihood function for the vector of parameters $[a, b, \theta]$ can be expressed as

$$\ln L = n\ln a + n\ln b + 2n\ln\theta - n\ln(\theta^2+1) + \sum_{i=1}^{n}\ln(\theta+x_i)$$
$$-\theta\sum_{i=1}^{n}X_i + (a-1)\sum_{i=1}^{n}\ln\left[1-(1+\frac{\theta x_i}{\theta^2+1})e^{-\theta x_i}\right]$$
$$+(b-1)\sum_{i=1}^{n}\ln\left\{1-\left[1-\left(1+\frac{\theta x_i}{\theta^2+1}\right)e^{-\theta x_i}\right]^a\right\} \qquad (23)$$

The components which corresponding to the parameters in ϕ are calculated by differentiating eq. (23) with respect to three parameters as follows.

$$\frac{\partial \ln L}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\ln\left[1-\left(1+\frac{\theta x_i}{\theta^2+1}\right)e^{-\theta x_i}\right]$$
$$-(b-1)\sum_{i=1}^{n}\frac{\left[1-\left(\frac{\theta x_i}{\theta^2+1}\right)e^{-\theta x_i}\right]^a\ln\left[1-\left(1+\frac{\theta x_i}{\theta^2+1}\right)e^{-\theta x_i}\right]}{\left\{1-\left[1-\left(1+\frac{\theta x_i}{\theta^2+1}\right)e^{-\theta x_i}\right]^a\right\}} \qquad (24)$$

$$\frac{\partial \ln L}{\partial b}$$

$$= \frac{n}{b} + \sum_{i=1}^{n} \ln \left\{ 1 - \left[1 - \left(1 + \frac{\theta x_i}{\theta^2 + 1} \right) e^{-\theta x_i} \right]^{a} \right\} \quad (25)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{2n}{\theta} - \frac{2n\theta}{\theta^2 + 1} + \sum_{i=1}^{n} \frac{1}{\theta + X_i} - \sum_{i=1}^{n} X_i$$

$$-(a-1) \sum_{i=1}^{n} \frac{X_i e^{-\theta x_i} \left[\frac{1-\theta^2}{(\theta^2+1)^2} - 1 - \frac{\theta^2 x_i}{\theta^2+1} \right]}{\left[1 - \left(1 + \frac{\theta x_i}{\theta^2+1} \right) e^{-\theta X} i \right]}$$

$$+a(b-1) \sum_{i=1}^{n} \frac{X_i e^{-\theta x_i} \left[\frac{1-\theta^2}{(\theta^2+1)^2} - 1 - \frac{\theta^2 x_i}{\theta^2+1} \right] \left[1 - \left(1 + \frac{\theta x_i}{\theta^2+1} \right) e^{-\theta x_i} \right]^{a-1}}{\left\{ 1 - \left[1 - \left(1 + \frac{\theta x_i}{\theta^2+1} \right) e^{-\theta x_i} \right]^{a} \right\}} \quad (26)$$

The maximum likelihood estimates (MLE_x) of the unknown parameters can be obtained by setting the system of non-linear equations24, 25, and 26 to zero and solving them simultaneously.Therefore, we have to use mathematical package like Mathcad to get the MLE_s of the unknown parameters.

4.2 Bayesian Estimation

If Ω is a vector of parameters and the prior is known that under a squared error loss function, Bayes estimate of any function of the parameters is the posterior expectation of that function. The estimator is given by its expectation with respect to the posterior density (Jeffreys, (1948)). The posterior of Ω is given by

$$P(\underline{\Omega} \setminus x) = \frac{\ell(x \setminus \underline{\Omega})\ P(\underline{\Omega})}{\int \ell(x \setminus \underline{\Omega})\ P(\underline{\Omega}) d\underline{\Omega}} \quad (27)$$

Where $\ell(\underline{x} \setminus \underline{\Omega})$ is the likelihood function, $P(\underline{\Omega})$ is the joint prior density of $\underline{\Omega}$.

Bayesian estimation is always given as a ratio of two integrals which, in most cases, can be expressed in numerical forms. Consequently an approximation is needed. In the following section we present an approximation due to Lindley (1980) by which this ratio can be approximated.

Assume that the prior distribution of a , b and θ are respectively

$$P\ (a) = \frac{1}{a}\ ,\ a > 0$$

$$P\ (b) = \frac{1}{b}\ ,\ b > 0$$

And $$P(\theta) = \frac{\beta^{\alpha}}{\Gamma\alpha}\ \theta^{\alpha-1}\ e^{-\beta\theta},\ \theta > 0\ ,\ \alpha,\beta > 0$$

Let the parameters a , b and θ are independent, then the joint prior of a , b and θ will be

$$P\ (a\ ,b,\theta) \propto a^{-1}\ b^{-1}\ \theta^{\alpha-1} e^{-\beta\theta}\ \ ,\ a,b > 0\ ,\ \theta,\alpha,\beta > 0 \quad (28)$$

Using equation (28), we have shown that the posterior density function is proportional to the product of the likelihood functions and prior. That is

P* ($\underline{\Omega}$/x) ∝ P ($\underline{\Omega}$) ℓ ($\underline{\Omega}$ \ x)

Where $\underline{\Omega}$ = (a, b , θ) and P ($\underline{\Omega}$) is the prior density and ℓ ($\underline{\Omega}$ \ x) is the likelihood function of the parameters based on the vector of observation x.

Let u ($\underline{\Omega}$) be any function of the vector of parameters $\underline{\Omega} = (\Omega_1, .., \Omega_m)$. Under the squared error loss function, the Bayesian estimation of u ($\underline{\Omega}$) is the mean of the posterior distribution, given by

$$\hat{u}(\underline{\Omega}) = E(u(\underline{\Omega}) \setminus x) = \frac{\int \dots \int u(\Omega)\ P^*(\underline{\Omega} \setminus x)\ d\Omega_1 \dots d\Omega_m}{\int \dots \int P^*(\underline{\Omega} \setminus x) d\Omega_1 \dots d\Omega_m} \quad (29)$$

4.2.1 Lindley Expansion

Lindley presented the following asymptotic expansion one way of obtaining

is by computing each of the integrals in the numerator and denominator and then finding the ratio. In general, both integrals should be computed numerically. Instead, Lindely integrals toobtain an approximate Bayesian estimate of u ($\underline{\Omega}$).

The approximate calculation of the ratio of integrals of the form (Lindley, 1980, Rashwan and Salem, 2014)

Posterior mean = $$\int u(\underline{\Omega}).e^{L(\underline{\Omega}\setminus x)+P(\underline{\Omega})} d\underline{\Omega} \Big/ \int\ e^{L(\underline{\Omega}\setminus x)+P(\underline{\Omega})} d\underline{\Omega}$$

Where

$\underline{\Omega} = (\Omega_1, .., \Omega_m)$is a vector of parameters

L ($\underline{\Omega}$/x) is the logarithm of the likelihood function

P ($\underline{\Omega}$) is the prior distribution for $\underline{\Omega}$

Let Q ($\underline{\Omega}$) = Log P ($\underline{\Omega}$) we obtain

The posterior expectation

$$E(u(\underline{\Omega}) \setminus x) = \int u(\underline{\Omega}).e^{L(\underline{\Omega}\setminus x)+Q(\underline{\Omega})} d\underline{\Omega} / \int e^{L(\underline{\Omega}\setminus x)+Q(\underline{\Omega})} d\underline{\Omega} \qquad (30)$$

The posterior expectation using Lindley expansion

$$E(u(\underline{\Omega}) \setminus x) = u + \frac{1}{2}\sum_{i,j}(u_{ij} + 2u_i Q_j)\sigma_{ij} + \frac{1}{2}\sum_{i,j,k,I} L_{ijk} + u_i \sigma_{ij}\sigma_{ki}$$

Where $E(u(\underline{\Omega}) \setminus x)$ is Bayesian estimator of u ($\underline{\Omega}$) under a squared error loss function and all summations run over all suffixes from 1 to m

Hence each suffix denotes differentiation once with respect to the variable having that suffix. Thus L222 is the third derivative with respect to Q2 similar notations are used for u and Q.

Note that:

u = u ($\underline{\Omega}$), $\quad u_i = \frac{\partial u}{\partial \Omega_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial \Omega_i \partial \Omega_j}, \quad Q_j = \frac{\partial Q}{\partial \Omega_j}, \quad L_{ijk} = \frac{\partial^3 L}{\partial \Omega_i \partial \Omega_j \partial \Omega_k}$

and σ_{ij} is the (i,j) the element in the inverse of matrix {-Lij}all evaluated at the maximum likelihood estimates of the parameters.

5. Applications

The Kw-Sh distribution has been fitted to two real lifetime data sets of COVID-19 mortality rates from Italy and Mexico [see https://covid19.who.int/].For more information about Covid-19 data see, Abdel-Rahman (2020).

The first data represents a COVID-19 mortality rates data belongs to Italy of 59 days, that is recorded from 27 February to 27 April 2020.

The data are as follows: 4.571 7.201 3.606 8.479 11.410 8.961 10.91910.908 6.503 18.474 11.010 17.337 16.561 13.226 15.137 8.69715.787 13.333 11.822 14.242 11.273 14.330 16.046 11.950 10.28211.775 10.138 9.037 12.396 10.644 8.646 8.905 8.906 7.407 7.4457.214 6.194 4.640 5.452 5.073 4.416 4.859 4.408 4.639 3.148 4.0404.253 4.011 3.564 3.827 3.134 2.780 2.881 3.341 2.686 2.814 2.5082.450 1.518.

The second data represents a COVID-19 mortality rate data belongs to Mexico of 108 days, that is recorded from 4 March to 20 July 2020.

This data formed of rough mortality rate. The data are as follows:
8.826 6.105 10.383 7.267 13.220 6.015 10.855 6.122 10.685 10.0355.242 7.630 14.604 7.903 6.327 9.391 14.962 4.730 3.215 16.49811.665 9.284 12.878 6.656 3.440 5.854 8.813 10.043 7.260 5.9854.424 4.344 5.143 9.935 7.840 9.550 6.968 6.370 3.537 3.286 10.1588.108 6.697 7.151 6.560 2.988 3.336 6.814 8.325 7.854 8.551 3.2283.499 3.751 7.486 6.625 6.140 4.909 4.661 1.867 2.838 5.392 12.0428.696 6.412 3.395 1.815 3.327 5.406 6.182 4.949 4.089 3.359 2.0703.298 5.317 5.442 4.557 4.292 2.500 6.535 4.648 4.697 5.459 4.1203.922 3.219 1.402 2.438 3.257 3.632 3.233 3.027 2.352 1.205 2.0773.778 3.218 2.926 2.601 2.065 1.041 1.800 3.029 2.058 2.326 2.5061.923.

The required numerical evaluations are implemented using Mathcad package software. In order to compare the KW-Sh distribution, Shanker and exponential distributions, Akaike information criterion (AIC) for two real data sets has been computed and presented in the following table (1).

Table (1). MLE_s of model's parameters and AIC statistic for real data sets

Data sets	Models	Parameters estimate			AIC
		θ	a	B	
1	Shanker	0.1663	-	-	547.4
	Exponential	0.2106	-	-	555.1
	KW-Sh	0.243	1.139	0.882	524.2
2	Shanker	0.1834	-	-	384.3
	Exponential	0.209	-	-	376.3
	KW-Sh	0.263	1.57	1.62	372.75

Note that, $AIC = -2\ln L + q$, where $\ln L$ denotes to the log- likelihood function and $q$is the number of parameters. From table (1), the results show that the KW-Sh distribution has the smaller value of AIC statistic when compared to that the value of the Shanker and exponential distributions. So, the KW-Shankermodel provides a better fit to these data.

Table (2): The estimation and the Variance of estimators

Parameters	Maximum Likelihood method		Bayesian method	
	Data set 1	Data set 2	Data set 1	Data set 2
A	1.139 (0.0206)	1.57 (0.007)	1.138 (0.0017)	1.75 (0.0029)
b	0.882 (0.0116)	1.62 (0.0114)	0.832 (0.0071)	1.82 (0.0091)
θ	0.243(0.0047)	0.263(0.0059)	0.321(0.00067)	0.189 (0.0051)

From table (2) Bayesian estimators have variance less than variance of maximum likelihood method, so the Bayesian estimation is better than the maximum likelihood estimation.

6. Conclusions

In this paper, a three-parameter lifetime distribution called the Kw-Sh distribution is introduced. Some statistical properties of this distribution such as the moments, moment generating function, incomplete moments, quantile function, entropies and mean deviation are derived and studied. We have presented two methods to estimate unknown parameters of K.W-SH distribution. Parameters of the KW-Sh distribution are estimated using the maximum likelihood estimation method and Bayesian estimation method. We used COVID-19 data sets to compare these methods. The results gives the best estimate of the Bayesian method because it have less variance. The proposed distribution is applied to COVID-19 data sets, the KW-Sh model gives a better fit than Shanker and exponential distributions.

References

[1] Abdel-Rahman, M. AM, (2020), "Academic Attitudes toward the role of Social Media in Shaping Electronic Public Opinion about Crises An applied Study on (Corona Virus Crisis)", Inf Sci LettVol. 9, No.2, pp.143-160.

[2] Aleem, M., Sufyan, M., Khan, N.S., and Ali, K., "Kumaraswamy double invers exponential (KW-DIE) distribution, In: proceedings of the 11th international conference on statistical sciences, vol. 25, pp. 93-104,2013.

Arnold B.C, Balakrishnan N. and Nagaraja, ,H.N.,(2008)," A First Course in Order Statistics", Siam Philadelphia

[3] Bourguignon, M., Silva, R.B., Zea, L. M., and Cordeiro, G.M., (2012),"The Kumaraswamy Pareto distribution", Journal of Statistical theory and application, vol. 12, pp. 129-144,DOI:10.2991.

[4] Carrasco, M.F., Ferrari, S. L., and Cordeiro, G.M.,(2010)," A New Generalized Kumaraswamy Distribution", arxiv:1004.09 IIVI (stat. ME), vol. 17.

[5] Cordeiro, G.M., and de Castro, M.,(2010),"A New Family of Generalized Distribution", Journal of statistical computation and Simulation, vol. 81, pp. 883-898,,DOI:10.1080

[6] Cordeiro, G.M., Ortega, E.M., and Nadarajah, S.,(2010),"The Kumaraswamy Weibull Distribution with Application To Failure Data", J Frankl Inst., vol. 347, pp. 1399-1429

[7] Cordeiro, G. M., Junior, A.C., Demetrio, C.G., Ortega, E.M., and Pescim, R.R., (2014),"Some New Results For The Kumaraswamy Modified Weibull Distribution", Journal of statistical theory and applications, vol, 13, pp. 88-104,DOI:10.2991.

[8] Cordeiro, G.M; Saboor, A., Khan, N.M., Ozel, G. and Pascoa, M.A.,(2016),” The Kumaraswamy Exponential Weibull: Theory and Applications", Hacettepe Journal of mathematics and statistics, vol. 45, pp. 1203-1229,DOI: 10.1155/6396.

[9] Elbatal, I, and Elgarhy, M., (2013),"Statistical Properties of Kumaraswamy Quasi Lindly Distribution", In T.J. Math Trends Technol., vol. 4, pp. 237-246.

[10] Fuller, E.J., Frieman, S., Quinn, G., and Carter, W. ,(1994)," Fracture Mechanics Approach To The Design of Glass Aircraft Windows: A Case Study", SPIE, proc. vol. 2286, pp. 419-430,DOI: 10.1117/12.187363.

[11] Ghitany, M. E., Atieh, B. and Nadarajah, S.,(2008)," Lindly Distribution aAnd Its Application", Mathematics computing and simulation, vol. 78, pp. 493-506, DOI:10.1061

[12] Gomes, A.E., da Silva, C. Q., Cordeiro, G.M., and Ortega, E. M.,(2014),"A new lifetime model: The Kumaraswamy generalized Rayleigh distribution", J. stat. comput. Simul. vol. 84, pp. 290-309,DOI: 10.1080/00949659608811741.

[13] Jeffreys, H.,"Theory of probability", second edition, Oxford, UK:Claredon Press,1948.

[14] Kumaraswamy, P.,(1980),"Generalized Probability Density Function for Double-Bounded Random Processes ", Journal of hydrology, vol. 46, pp. 79-88, DOI:10.1016.

[15] Lindley, D.V.,(1980), "Approximate Bayesian Method", Trabajos de Esta- distica de Investigacion Operativa, Vol 31, PP: 223-245..

[16] Mansouir, M. M., Aryal, G., Afify, A. Z., and Ahmad, M.,"(2018),"The Kumaraswamy Exponentiated FréchetDistribution", Pak. J. Statist., vol. 34, pp. 177-193,.

[17] Nadarajah, S., Cordeiro, G. M., and Ortega, E.M.,(2011),"General Results for the Kumaraswamy- G Distribution", Journal of statistical computation, and Simulation,doi: 10.1080/0094655.2011.562504.

[18] Rashwan, N.I., (2016), A note on Kumaraswamy exponentiated Rayleigh distribution, Journal of statistical theory and applications, vol., 15, pp. 286-295,2016.

[19] Rashwan, N.I. and Salem H. A. , A,(2014)," Bayesian method to estimate the parameters of regression model". , Advances and applications in statistics, vol. 43, pp, 37-52.

[20] Shanker, R., (2015),Shanker distribution and its applications international journal of statistics and applications, vol. 5, pp. 338-348.

[21] Shannon, C.E., (1948), A mathematical theory of communication,Bell Syst. Tech. J.,vol27,379-423